Linear Mass Commutator Calculation

Jay R. Yablon – June 19, 2008

I. Known Square Mass Commutation Calculation

Consider a particle of mass *m* as a single particle system. Consider canonical coordinates x_μ, and that at least the space coordinates $x_j;\ j=1,2,3$ are operators. If we require that the mass *m* must commute with all operators, then we must have $[x_\mu, m]=0$, and by easy extension, $[x_\mu, m^2]=0$. It is well known that the commutation condition $[x_\mu, m^2]=0$, taken together with the on-shell mass relationship $m^2 = p^\sigma p_\sigma$ and the single-particle canonical commutation relationship $[x_j, p_k]=i\eta_{jk};\ j,k=1,2,3$, where $diag(\eta_{\mu\nu})=(-1,+1,+1,+1)$ is the Minkowski tensor, leads inexorably to the commutation relationship:

$$[x_k, p_0] = -ip_k / p^0 = -iv_k \qquad (1.1)$$

where v_k is the particle velocity (in *c*=1 units) along the *k*th coordinate. I leave the detailed calculation as an exercise for the reader not familiar with this calculation, and refer also to the sci.physics.research thread at http://www.physicsforums.com/archive/index.php/t-142092.html or http://groups.google.com/group/sci.physics.research/browse_frm/thread/d78cbfecf703ff6a#.

I would ask for your comments on the following calculation, which is totally analogous to the calculation that leads to (1.1), but which is done using the linear mass *m* rather than the square mass m^2, and using the Dirac equation written as $m\psi = \gamma^\nu p_\nu \psi$, in lieu of what is, in essence, the Klein Gordon equation $m^2\phi = p^\sigma p_\sigma \phi$ that leads to (1.1).

2. Maybe New?? Linear Mass Commutation Calculation

Start with Dirac's equation written as:

$$m\psi = \gamma^\nu p_\nu \psi \ . \quad (2.1)$$

Require that:

$$[x_\mu, m] = 0 \qquad (2.2)$$

Continue to use the canonical commutator $[x_j, p_k] = ig_{jk}$ of (1.3). Multiply (2.1) from the left by x_μ noting that $[\gamma^\nu, x_\mu]=0$ to write:

$$x_\mu m\psi = \gamma^\nu x_\mu p_\nu \psi = \gamma^0 x_\mu p_0 \psi + \gamma^j x_\mu p_j \psi \quad . \quad (2.3)$$

This separates into:

$$\begin{cases} x_0 m\psi = \gamma^0 x_0 p_0 \psi + \gamma^j x_0 p_j \psi \\ x_k m\psi = \gamma^0 x_k p_0 \psi + \gamma^j x_k p_j \psi \end{cases} . \quad (2.4)$$

Now, use the canonical relation $\left[x_j, p_k\right] = i\eta_{jk}$ to commute the space (k) equation, thus:

$$\begin{aligned} x_k m\psi &= \gamma^0 x_k p_0 \psi + \gamma^j x_k p_j \psi = \gamma^0 x_k p_0 \psi + \gamma^j \left(p_j x_k + i\eta_{jk}\right)\psi \\ &= \gamma^0 x_k p_0 \psi + \gamma^j p_j x_k \psi + i\gamma_k \psi \\ &= \gamma^0 x_k p_0 \psi + m x_k \psi - \gamma^0 p_0 x_k \psi + i\gamma_k \psi \end{aligned} \quad . \quad (2.5)$$

In the final line, we use Dirac's equation written as $m x_\mu \psi = \gamma^\nu p_\nu x_\mu \psi = \gamma^0 p_0 x_\mu \psi + \gamma^j p_j x_\mu \psi$, and specifically, the $\mu = k$ component equation $\gamma^j p_j x_k \psi = m x_k \psi - \gamma^0 p_0 x_k \psi$.

If we require that $\left[x_\mu, m\right] = 0$, which is (2.2), then (2.5) reduces easily to:

$$\gamma^0 \left[x_k, p_0\right]\psi = -i\gamma_k \psi \quad , \quad (2.6)$$

Finally, multiply from the left by γ^0, and employ $\gamma^0 \gamma_k \equiv \alpha_k$ and $\gamma^0 \gamma^0 = 1$ to write:

$$\boxed{\left[x_k, p_0\right]\psi = -i\alpha_k \psi} \quad . \quad (2.7)$$

If we contrast (2.7) to (1.1) written as $\left[x_k, p_0\right]\phi = -iv_k \phi$, we see that the velocity $p_k / p^0 = v_k$ has been replaced by the Dirac operator α_k, that is, $v_k \to \alpha_k$.

3. Questions

Here are my first set of questions:

1) Is the calculation leading to (2.7) correct, and is (2.7) a correct result, or have I missed something along the way?

2) If (2.7) is correct, has anyone seen this result before? If so where?

3) Now use the plane wave $\psi = ue^{ip^\sigma x_\sigma}$ so that we can work with the Dirac spinors $u\left(p^\mu\right)$, and rewrite (2.7) as:

$$\begin{cases} (\alpha_k - \lambda)u = 0 \\ \lambda = i\left[x_k, p_0\right] \end{cases} \quad (3.1)$$

The upper member of (3.1) is an eigenvalue equation. Reading out this equation, I would say that the commutators $\lambda = i[x_k, p_0]$ are the eigenvalues of the Dirac α_k matrices, which are:

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \text{ and } \boldsymbol{\alpha} = \begin{pmatrix} -\boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \quad (3.2)$$

in the respective Pauli/Dirac and Weyl representations, and that the *u* are the eigenvectors associated with these eigenvalues $\lambda = i[x_k, p_0]$. Am I wrong? If not, how would one interpret this result? Maybe the commutators $[x_j, p_k] = i\eta_{jk}$ can be discussed in the abstract, but it seems to me that the commutators $\lambda = i[x_k, p_0]$ can only be discussed as the eigenvalues of the matrices α_k with respect to the eigenstate vectors *u*. This, it seems, would put canonical commutation into a somewhat different perspective than is usual.

Just as Dirac's equation reveals some features that cannot be seen strictly from the Klein Gordon equation, the calculation here seems to reveal some features about the canonical commutators that the usual calculation based on $[x_\mu, m^2] = 0$ and $m^2 = p^\sigma p_\sigma$ cannot, by itself, reveal.

I'd appreciate your thoughts on this, before I proceed downstream from here.

Thanks,

Jay.