Lab Note 2, Part 2: Gravitational and Inertial Mass, and Electromagnetism as Geometry, in 5-Dimensional Spacetime

Jay R. Yablon (jyablon@nycap.rr.com), February 6, 2008

Note: This includes Lab Note 2, Part 3, dated February 14, 2008, beginning on page 17

1. Introduction

It has been understood at least since Galileo's refutation of Aristotle which legend situates at the Leaning Tower of Pisa, that heavier masses and lighter masses similarly-disposed in a gravitational field will accelerate at the same rate and reach the ground after identical times have elapsed. Physicists have come to describe this with the principle that the "gravitational mass" and the "inertial mass" of any material body are "equivalent." As a material body becomes more massive and so more-susceptible to the pull of a gravitational field (back when gravitation was viewed as action at a distance), so too this increase in massiveness causes the material body in equal measure to resist the gravitational pull. By this equivalence, the result is a "wash," and so with the neglect of any air resistance, all the bodies accelerate and fall at the same rate. (The other consequence of Galileo's escapade, is that it strengthened the role of experimental testing, in relation to the "pure thought" upon which Aristotle had relied to make the "obvious" but untested and in fact false argument that heavy objects should fall faster. In this way, it spawned the essence of what we today know as the scientific method which remains a dynamic blend of thought and creativity, with experience and cold, hard numbers derived from measurement of masses, lengths, and times.)

Along his path to developing the General Theory of Relativity (GTR), Albert Einstein made a brief stop in 1911 in an imaginary elevator, to conduct a *gedanken* in which he concluded that the physical experience of an observer falling freely in a gravitational field before terminally hitting the ground is no different from what was commonly thought of as Newton's inertial motion in which a body in motion remained in motion unless acted upon by a "force." (GTR later showed that this was not quite true, the "asterisk" to this insight arising from the so-called tidal forces.) And, he concluded that the force one feels standing on the floor of an elevator in free fall to which a constant force is then applied, is no different from the force one feels when standing on the surface of the earth.

The General Theory of Relativity, in the end, captured inertial motion and its close cousin of free-fall motion in a gravitational field, in the most elegant way, as simple geodesic motion in a curved geometry along geodesic paths which coincide precisely with the paths one observes for bodies moving under gravitational influences. This was a triumph of the highest order, as it placed gravitational theory on the completely-solid footing of Riemannian geometry, and became the "gold standard" against which all other physical theories are invariably measured, even to this day. ("Marble and wood" is another oft-employed analogy.)

However, the question of "absolute acceleration," that is, of an acceleration which is not simply a geodesic phenomenon of unimpeded free fall through a swathe carved out by geometry, but rather one in which an observer actually "feels" a "force" which can be measured by a "weight scale" in physical contact between the observer and that body which applies the force, is in fact not resolved by GTR. To this day, it is hotly-debated whether or not there is such a thing as "absolute acceleration." Surely, the forces we feel on our bodies in elevators and cars and standing on the ground are real enough, but the question is whether there is some way to understand these forces – which are impediments to what would otherwise be our own geodesic free fall motion in spacetime under the influence of gravity and nothing more – *as geodesic forces in their own right*, simply of a different, supplemental, and perhaps more-subtle character than the geodesics of gravitation. That is the central question to be examined in this lab note.

If we think of the "gravitational mass" of a material body more generally as its "interaction mass" for the specific circumstance in which the "interaction" is "gravitational," then the answer to the question whether the real forces we feel when our bodies are "absolutely" accelerated might still be described in terms of geometric geodesics, may still lurk amidst Galileo's legendary escapade at Pisa, but with a twist. In this situation, *the "interaction mass" of a material body is now* ***inequivalent*** *to its "inertial mass,"* because that interaction is now "electrical" rather than "gravitational." Here, Aristotle has his day, because "electrically-heavier" bodies do fall faster than "electrically-lighter" ones.

How do electrical masses now come into play? When we fail to maintain our gravitational geodesic motion by failing to morph through the floor of the elevator, or when we fail to continue our gravitational free fall by not falling unimpeded through the earth's surface, it is because we are stopped by the collective electrical repulsion between billions of electrons in our bodies and billions more in the elevator floor or the earth's ground. It is because the electrical interaction has now trumped the gravitational interaction and taken us off of our gravitational geodesics. Or, perhaps, if we can obtain a geometric insight into electrodynamics, it is because we are now leaving the gravitational geodesic, and the atoms in our body are instead embarking upon a different sort of geodesic path which now coincides with the path that has long been observed as the Lorentz force motion of a charged mass in an electromagnetic field.

In light of the quantum revolution of the 20th century, one other consideration is in order. In this discussion, we are talking not about quantum phenomenon, but about bulk phenomenon which lend themselves to completely classical description. In the same way that a bulk material body follows a geodesic path through gravitation, the question we raise is whether bulk electrical bodies, or large numbers of electrons in an electric field, can also be understood, via their Lorentz force motion, to be following geodesic paths made of "marble" no less fine than the marble with which General Relativity directs the paths of material bodies through spacetime geometry in a manner that coincides precisely with what we observe and measure to be a gravitational path. We want to understand why we don't fall through the elevator or through the earth. Not only do we want to understand this in a way that avoids contradicting the geodesic principles of gravitation, we want to do so in a way that seamlessly extends these principles in a totally-self consistent way, into the electromagnetic arena.

Five-dimensional theories (or higher), have frequently been a foundation upon which to try to merge classical gravitation with classical electrodynamics. Kaluza and Klein began the trend, Einstein looked favorably on the effort, many others have followed, but to this day, there is as yet no theory which has been fully compelling in all aspects, and which at the same time, is motivated to a fifth dimension in a completely natural way, conservatively based on solid principles of observational physics which are already firmly-established.

2. Using Dirac's "Gamma-5" to Motivate a Fifth, Timelike Dimension

One of the most important connections in all of physics is given by the Dirac relationship:

$$\tfrac{1}{2}\left\{\gamma^{\mu}\gamma^{\nu}+\gamma^{\nu}\gamma^{\mu}\right\}\equiv\eta^{\mu\nu}, \qquad (2.1)$$

whereby the Dirac γ^{μ} matrices, $\mu=0,1,2,3$, are *defined* so as to reproduce the Minkowski metric tensor $\mathrm{diag}\left(\eta^{\mu\nu}\right)=\left(+1,-1,-1,-1\right)$ under anticommutation. This relationship not only underlies

Dirac's equation, but also ensures that the Klein-Gordon equation applies to fermions as well as bosons. It is firmly established in all respects, and certainly must be regarded as one of those physical relationships which is made of "marble" over wood.

Also made of "marble," is the axial Dirac matrix first motivated by Weyl:

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 , \tag{2.2}$$

which is *defined* from matrix-multiplying the other four Dirac matrices, and which has a well-established and rigorously-observed physical meaning in relation to the left- and right-chiral handedness of elementary fermions. We know that when the γ^μ are sandwiched between Dirac spinors in the form $j^\mu = \overline{\psi}\gamma^\mu\psi$, the resulting current source density j^μ (also thought of as a probability and flux density) transforms as a four-vector in spacetime. We also know that $j^5 \equiv \overline{\psi}\gamma^5\psi$ is a "pseudo-scalar." Though there are five such Dirac gamma matrices, only four these are multiplicatively independent.

We will now motivate a five-dimensional spacetime, based on (2.1) and (2.2), in the following way: The five-dimensional spacetime we employ will be one which is *defined* as a geometry in which j^μ and j^5, taken together, all transform together as a five-vector $j^{\mathrm{M}} \equiv (j^\mu, j^5)$, with $\mathrm{M} = 0,1,2,3,5$. In this five-dimensional space, we employ uppercase Greek indexes. We maintain the lower case $\mu = 0,1,2,3$ for the usual 4-dimensional spacetime subspace.

The metric tensor for such a five-dimensional geometry, must therefore be formed from the anticommutator of all five of the γ^{M}, similarly to (2.1). That is, if $j^{\mathrm{M}} \equiv (j^\mu, j^5)$ is to transform as a five-vector, then we must define a five-dimensional, 5x5 Minkowski metric tensor according to:

$$\eta^{\mathrm{MN}} \equiv \tfrac{1}{2}\left\{\gamma^{\mathrm{M}}\gamma^{\mathrm{N}} + \gamma^{\mathrm{N}}\gamma^{\mathrm{M}}\right\}. \tag{2.3}$$

Given the well-known anticommutation properties of the five γ^{M}, one can readily deduce that $\mathrm{diag}(\eta^{\mathrm{MN}}) = (+1,-1,-1,-1,+1)$, and that $\eta^{\mathrm{MN}} = 0$ for $\mathrm{M} \neq \mathrm{N}$. The usual Minkowski metric tensor $\eta^{\mu\nu}$ is of course preserved in the 16 = 4x4 components of η^{MN} for which $\mathrm{M},\mathrm{N} = \mu,\nu = 0,1,2,3$. Importantly, because $\eta^{55} = +1$, we find that this fifth dimension has a *timelike*, rather than a spacelike signature. Put succinctly: *this five-dimensional geometry consists of two timelike and three spacelike dimensions*.

We next define infinitesimal coordinate intervals in the usual way, including a fifth dx^5 interval, that is, $dx^{\mathrm{M}} \equiv \left(dx^0, dx^1, dx^2, dx^3, dx^5\right)$. Because γ^5 is known as the "axial" matrix and because it is associated with a timelike metric signature as noted just above, we shall refer to x^5 as the "axial time" coordinate, and will continue to refer to x^0 as the "ordinary time" coordinate. The x^1, x^2, x^3 coordinates of course retain their role as ordinary space coordinates.

Geometrically, in light of the two timelike dimensions, it will often be very useful to regard time not as a "time line" but as a "*time plane*." Thus, following Feynman, we might not only think about worldlines which move forwards and backwards in time, but also which move *sideways* in time, and at various angles through the time plane. In fact, it is particularly helpful if one draws a vertical coordinate axis for ordinary time x^0 orthogonal to a horizontal coordinate axis for axial time x^5, to represent the "time plane." Then for material bodies "at rest," $dx^1 = dx^2 = dx^3 = 0$, one may speak about the "angle" at which their worldlines move through this time plane. As we shall see, this may lead to a solely-geometric way to understand rest mass, electric charge, and electrical Lorentz Force motion, as geodesic motion through curved, non-Euclidean geometry.

The next step is to specify a metric interval $d\mathrm{T}$ for this five-dimensional spacetime. One might regard this as a "flat" spacetime and so define $d\mathrm{T}^2 \equiv \eta^{\mathrm{MN}} dx_{\mathrm{M}} dx_{\mathrm{N}}$. However, if our objective is to understand the motions of electrical bodies on the basis of geodesic paths through a geometry, we must take one final step, and allow this five-dimensional geometry to be a curved, non-Euclidean geometry just like that which is used in GTR. Thus, we shall establish a metric tensor $g_{\mathrm{MN}} = \eta_{\mathrm{MN}} + \kappa h_{\mathrm{MN}}$ just as in GTR, and specify the weak-field limit according to $g_{\mathrm{MN}} \to \eta_{\mathrm{MN}}$, i.e., $h_{\mathrm{MN}} \to 0$. We further maintain the usual interval in the 4-dimensional spacetime subspace, using $d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$, and we thereby specify metric intervals in this 5-dimensional spacetime with axial time, according to:

$$\begin{aligned} d\mathrm{T}^2 \equiv g_{\mathrm{MN}} dx^{\mathrm{M}} dx^{\mathrm{N}} &= g_{\mu\nu} dx^\mu dx^\nu + g_{5\nu} dx^5 dx^\nu + g_{\mu 5} dx^\mu dx^5 + g_{55} dx^5 dx^5 \\ &= d\tau^2 + g_{5\nu} dx^5 dx^\nu + g_{\mu 5} dx^\mu dx^5 + g_{55} dx^5 dx^5 \end{aligned} . \tag{2.4}$$

(As an aside, for completeness, having extended $\eta_{\mathrm{MN}} \to g_{\mathrm{MN}}$ to incorporate curvature and hence gravitation, we should also return to (2.3), and redefine the Dirac Gamma matrices so as to incorporate these curvatures as well. Thus, we now *define* a new set of Dirac matrices $\Gamma^{\mathrm{M}}\left(x^{\mathrm{M}}\right)$ from the contravariant $g^{\mathrm{MN}}\left(x^{\mathrm{M}}\right)$, according to:

$$\tfrac{1}{2}\left\{\Gamma^{\mathrm{M}}\Gamma^{\mathrm{N}} + \Gamma^{\mathrm{N}}\Gamma^{\mathrm{M}}\right\} \equiv g^{\mathrm{MN}}. \tag{2.5}$$

These Γ^{M}, which are now fields rather than constant matrices, and which approach the usual γ^{M} in the weak-field limit, now implicitly include gravitational effects. When employed in Dirac's equation, these Γ^{M} lead to some very interesting ways to interpret the Schwinger magnetic moments as indicative of gravitational effects near the Planck length which give clues as to the true "size" of the elementary fermions, and these may also bear a relationship to the $\Gamma^{\mu} = \gamma^{\mu} + \Lambda^{\mu}$ used in perturbation theory to represent non-divergent perturbative corrections. But these are topics for an entirely different paper. Let's return to the main thread of discussion by returning to (2.4).)

3. A Possible Geometric Interpretation of Rest Mass

In applying (2.4), we extend all of the customary GTR relationships from four to five dimensions. Thus, $g_{\mathrm{MN}} = g_{\mathrm{NM}}$ is a symmetric tensor, inverses are specified by $g^{\mathrm{MN}} g_{\mathrm{N\Sigma}} = \delta^{\mathrm{M}}{}_{\Sigma}$ (thus the g_{MN} and g^{MN} are used to lower and raise indexes), the covariant derivative of the metric tensor is defined by $g_{\mathrm{MN;\Sigma}} \equiv 0$, the 5-D Christoffel connections are $\Gamma^{\mathrm{M}}{}_{\Sigma\mathrm{T}} = \tfrac{1}{2} g^{\mathrm{MA}}\left(g_{\mathrm{A\Sigma,T}} + g_{\mathrm{TA,\Sigma}} - g_{\mathrm{\Sigma T,A}}\right)$, hence $\Gamma^{\mathrm{M}}{}_{\Sigma\mathrm{T}} = \Gamma^{\mathrm{M}}{}_{\mathrm{T}\Sigma}$, and the covariant derivative of a first rank vector A^{M} is $A^{\mathrm{M}}{}_{;\Sigma} = A^{\mathrm{M}}{}_{,\Sigma} + \Gamma^{\mathrm{M}}{}_{\mathrm{A}\Sigma} A^{\mathrm{A}}$.

Now, let's use algebraic manipulation to rewrite (2.4) in both of the following forms:

$$\frac{d\tau^2}{d\mathrm{T}^2} = 1 - g_{55}\frac{dx^5}{d\mathrm{T}}\frac{dx^5}{d\mathrm{T}} - g_{5\nu}\frac{dx^5}{d\mathrm{T}}\frac{dx^\nu}{d\mathrm{T}} - g_{\mu 5}\frac{dx^\mu}{d\mathrm{T}}\frac{dx^5}{d\mathrm{T}} \quad \text{and} \tag{3.1}$$

$$0 = g_{\mu\nu}\frac{dx^\mu}{d\mathrm{T}}\frac{dx^\nu}{d\mathrm{T}} - \left(1 - g_{5\nu}\frac{dx^5}{d\mathrm{T}}\frac{dx^\nu}{d\mathrm{T}} - g_{\mu 5}\frac{dx^\mu}{d\mathrm{T}}\frac{dx^5}{d\mathrm{T}} - g_{55}\frac{dx^5}{d\mathrm{T}}\frac{dx^5}{d\mathrm{T}}\right). \tag{3.2}$$

Using a velocity four-vector $u^\mu \equiv dx^\mu / d\tau$, these can be combined to obtain:

$$0=\frac{d\tau^2}{d\mathrm{T}^2}\left(g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}-1\right)=\frac{d\tau^2}{d\mathrm{T}^2}\left(u^\mu u_\mu-1\right). \qquad (3.3)$$

If we define a momentum four-vector $p^\mu \equiv mu^\mu$ for a mass *m* in the usual way, and contrast (3.3) to the equation $p^\mu p_\mu - m^2 = m^2\left(u^\mu u_\mu - 1\right) = 0$ for the energy-momentum of an on-shell mass m, we see that $d\tau / d\mathrm{T}$ in (3.3) plays a role identical to that of the mass m in $m^2\left(u^\mu u_\mu - 1\right) = 0$. In the mass shell equation, *m* is of course introduced "by hand," because the most one can deduce from the four-dimensional metric equation $d\tau^2 = g_{\mu\nu}dx^\mu dx^\nu$ is $u^\mu u_\mu - 1 = 0$; then we need to multiply through by a mass *m* which we simply take out of "thin air" based on our empirical knowledge that masses exist in nature. In contrast, in (3.3), the $d\tau / d\mathrm{T}$ multiplier of $u^\mu u_\mu - 1 = 0$ arises totally out of the five-dimensional geometry, with nothing introduced "by hand." $d\tau / d\mathrm{T}$ is marble, and *m* is wood.

We can capture this very-telling correspondence, by writing:

$$m \propto \frac{d\tau}{d\mathrm{T}}. \qquad (3.4)$$

That is, in some way to be further determined, the rest mass of a material body appears to be proportional to the ratio of $d\tau$ to $d\mathrm{T}$, and so may have a simple geometric foundation based on the trajectory of a worldline in the $x^0 - x^5$ time plane. Now, we are ready to examine the geodesics of this five-dimensional geometry.

4. The Geodesic Equation in Five Dimensions

The five-dimensional calculation to follow is analogous to one way of deriving the geodesic equation in four dimensions, though we will in any event carry out the calculation in full a) to help the reader review the basic role of the metric equation as the first integral of the equation of motion b) to convince the reader that the 5-D geodesic equation is in fact correct, and c) to establish a careful discipline about working in five dimensions, making sure that all of our calculations are carried out in a 5-covariant rather than only a 4-covariant manner.

We return to the metric equation (2.4), and now rewrite this, again via trivial algebraic rearrangement, as:

$$1 = g_{\mathrm{MN}} \frac{dx^{\mathrm{M}}}{d\mathrm{T}} \frac{dx^{\mathrm{N}}}{d\mathrm{T}} . \tag{4.1}$$

Taking the covariant derivative of each side, employing $g_{\mathrm{MN};\Sigma} = 0$, renaming indexes, commuting, and using $g_{\mathrm{MN}} = g_{\mathrm{NM}}$, enables us to then write:

$$0 = g_{\mathrm{MN}} \left(\frac{dx^{\mathrm{M}}}{d\mathrm{T}} \right)_{;\Sigma} \frac{dx^{\mathrm{N}}}{d\mathrm{T}} . \tag{4.2}$$

The covariant derivative $\left(\frac{dx^{\mathrm{M}}}{d\mathrm{T}} \right)_{;\Sigma} = \left(\frac{dx^{\mathrm{M}}}{d\mathrm{T}} \right)_{,\Sigma} + \Gamma^{\mathrm{M}}{}_{\mathrm{A}\Sigma} \left(\frac{dx^{\mathrm{A}}}{d\mathrm{T}} \right)$, which we substitute into (4.2). After some rearranging of terms, and assuming that $g_{\mathrm{MN}} \neq 0$, we can write:

$$0 = \frac{d}{dx^{\Sigma}} \frac{dx^{\mathrm{N}}}{d\mathrm{T}} \frac{dx^{\mathrm{M}}}{d\mathrm{T}} + \Gamma^{\mathrm{M}}{}_{\mathrm{A}\Sigma} \frac{dx^{\mathrm{A}}}{d\mathrm{T}} \frac{dx^{\mathrm{N}}}{d\mathrm{T}} . \tag{4.3}$$

Finally, we contract the Σ and N indexes, do some index renaming, and obtain the geodesic equation in 5-Dimensions:

$$\frac{d^2 x^{\mathrm{M}}}{d\mathrm{T}^2} + \Gamma^{\mathrm{M}}{}_{\Sigma\mathrm{T}} \frac{dx^{\Sigma}}{d\mathrm{T}} \frac{dx^{\mathrm{T}}}{d\mathrm{T}} = 0 . \tag{4.4}$$

This looks just like the four-dimensional equation, but for the indexes summing over all five dimensions rather than four, and the $d\mathrm{T}$ rather than $d\tau$ in the denominators. But, we can multiply through fully by $d\mathrm{T}^2 / d\tau^2$, and rewrite (4.4) as:

$$\frac{d^2 x^{\mathrm{M}}}{d\tau^2} + \Gamma^{\mathrm{M}}{}_{\Sigma\mathrm{T}} \frac{dx^{\Sigma}}{d\tau} \frac{dx^{\mathrm{T}}}{d\tau} = 0 . \tag{4.5}$$

Now, let us work with (4.5) above.

Equation (4.5) is a set of *five* independent equations. Let's separate it into the four spacetime equations represented by:

$$\frac{d^2 x^{\mu}}{d\tau^2} + \Gamma^{\mu}{}_{\Sigma\mathrm{T}} \frac{dx^{\Sigma}}{d\tau} \frac{dx^{\mathrm{T}}}{d\tau} = 0 \tag{4.6}$$

and the single axial-dimension equation:

$$\frac{d^2x^5}{d\tau^2}+\Gamma^5{}_{\Sigma\mathrm{T}}\frac{dx^\Sigma}{d\tau}\frac{dx^\mathrm{T}}{d\tau}=0. \tag{4.7}$$

Equation (4.6) is for $\frac{d^2x^\mu}{d\tau^2}$, which is the observed acceleration of a worldline in the observed four dimensions of spacetime. But, the five-dimensional summation in $\Gamma^\mu{}_{\Sigma\mathrm{T}}\frac{dx^\Sigma}{d\tau}\frac{dx^\mathrm{T}}{d\tau}$ in (4.6) adds some new terms to the usual gravitational geodesic equation. Specifically, (4.6) expands to:

$$\frac{d^2x^\mu}{d\tau^2}+\Gamma^\mu{}_{\sigma\tau}\frac{dx^\sigma}{d\tau}\frac{dx^\tau}{d\tau}+\Gamma^\mu{}_{\sigma5}\frac{dx^\sigma}{d\tau}\frac{dx^5}{d\tau}+\Gamma^\mu{}_{5\tau}\frac{dx^5}{d\tau}\frac{dx^\tau}{d\tau}+\Gamma^\mu{}_{55}\frac{dx^5}{d\tau}\frac{dx^5}{d\tau}=0 \tag{4.8}$$

which on account of $\Gamma^\mathrm{M}{}_{\Sigma\mathrm{T}}=\Gamma^\mathrm{M}{}_{\mathrm{T}\Sigma}$, we consolidate with some index renaming to:

$$\frac{d^2x^\mu}{d\tau^2}+\Gamma^\mu{}_{\sigma\tau}\frac{dx^\sigma}{d\tau}\frac{dx^\tau}{d\tau}+2\Gamma^\mu{}_{\tau5}\frac{dx^5}{d\tau}\frac{dx^\tau}{d\tau}+\Gamma^\mu{}_{55}\frac{dx^5}{d\tau}\frac{dx^5}{d\tau}=0 \tag{4.9}$$

5. Geodesic Motion of a Charged Mass in an Electromagnetic Field

Now, it is time to contrast (4.9) above to the Lorentz Force Law when taken together with the gravitational geodesic equation. This is, for example, set forth in equation (20.41) of *Gravitation* by Misner, Wheeler and Thorne:

$$\frac{d^2x^\mu}{d\tau^2}+\Gamma^\mu{}_{\sigma\tau}\frac{dx^\sigma}{d\tau}\frac{dx^\tau}{d\tau}-\frac{q}{m}F^\mu{}_\tau\frac{dx^\tau}{d\tau}=0. \tag{5.1}$$

At the outset, let us set aside the term $\Gamma^\mu{}_{55}\frac{dx^5}{d\tau}\frac{dx^5}{d\tau}$ in (4.9). We can do, this, for example, by setting $\Gamma^\mu{}_{55}=0$. We also move the $dx^5/d\tau$ in front of the $\Gamma^\mu{}_{\tau5}$, so that (4.9) now becomes:

$$\frac{d^2x^\mu}{d\tau^2}+\Gamma^\mu{}_{\sigma\tau}\frac{dx^\sigma}{d\tau}\frac{dx^\tau}{d\tau}+2\frac{dx^5}{d\tau}\Gamma^\mu{}_{\tau5}\frac{dx^\tau}{d\tau}=0 \tag{5.2}$$

Now, we contrast (5.1) directly with (5.2). The term $2\frac{dx^5}{d\tau}\Gamma^\mu{}_{\tau5}\frac{dx^\tau}{d\tau}$ in (5.2) is the "marble" of geometry, while the corresponding term $-\frac{q}{m}F^\mu{}_\tau\frac{dx^\tau}{d\tau}$ is the "wood" of an empirically-derived relationship. However, comparing these terms, we can give a totally geometric footing to the Lorentz force law if we note the proportionalities:

$$-\frac{q}{m} \propto \frac{dx^5}{d\tau}, \text{ and} \tag{5.3}$$

$$F^{\mu}{}_{\tau} \propto 2\Gamma^{\mu}{}_{\tau 5}. \tag{5.4}$$

Combining (3.4), $m \propto d\tau / d\mathrm{T}$, with (5.3), additionally yields:

$$-q \propto \frac{dx^5}{d\mathrm{T}} \tag{5.5}$$

In fact, if we substitute the proportionalities in (5.3) and (5.4) into (5.1) *as if* they were equalities, then the geodesic equation (5.2), totally-based in geometry, *is the Lorentz force law*. *Lorentz force motion now appears to be motion along a geodesic, just like gravitational motion*. Additionally, this geodesic motion appears to introduce an absolute acceleration, and hence "force," because of the *inequality* of electrical and inertial mass, which via (5.3) has its origins in the *geometric* statement $dx^5 \neq d\tau$. This *geometric* statement, is now at the root of the "force," and hence "absolute acceleration," experienced by a charged mass in an electromagnetic field. Yet, this force and absolute acceleration, still is a form of geodesic motion described by (5.2), and so rests fully upon the "marble" of geodesic motion through curved geometry.

In fact, let's take this geometric understanding of inertial mass and electric charge even a step further. Consider a material body viewed at rest, by setting $dx^1 = dx^2 = dx^3 = 0$, hence $d\tau = \sqrt{g_{00}}\, dx^0$. Then, according to (2.4):

$$d\mathrm{T}^2 \equiv g_{\mathrm{MN}} dx^{\mathrm{M}} dx^{\mathrm{N}} = g_{00} dx^0 dx^0 + g_{50} dx^5 dx^0 + g_{05} dx^0 dx^5 + g_{55} dx^5 dx^5. \tag{5.6}$$

Further, take the weak field approximation where $g_{\mathrm{MN}} \approx \eta_{\mathrm{MN}}$, $d\tau \approx dx^0$. Now, (5.6) becomes:

$$d\mathrm{T}^2 \approx \left(dx^0\right)^2 + \left(dx^5\right)^2. \tag{5.7}$$

Draw x^0 and x^5 as orthogonal axes, where x^0 is vertical and x^5 is horizontal (sideways axial time). From (5.7), $d\mathrm{T}$ is clearly the hypotenuse. Define an angle Θ such that $\tan\Theta \equiv dx^0 / dx^5$.

Now, in this weak field $g_{\mathrm{MN}} \approx \eta_{\mathrm{MN}}$ rest frame $dx^1 = dx^2 = dx^3 = 0$, we can write (3.4), (5.3), and (5.5), respectively, as:

$$m \propto \frac{d\tau}{d\mathrm{T}} \approx \frac{dx^0}{d\mathrm{T}} \approx \frac{dx^0}{\sqrt{\left(dx^0\right)^2 + \left(dx^5\right)^2}} = \sin\Theta . \tag{5.8}$$

$$-\frac{q}{m} \propto \frac{dx^5}{d\tau} \approx \frac{dx^5}{dx^0} \equiv \cot\Theta \tag{5.9}$$

$$-q \propto \frac{dx^5}{d\mathrm{T}} \approx \frac{dx^5}{\sqrt{\left(dx^0\right)^2 + \left(dx^5\right)^2}} = \cos\Theta \tag{5.10}$$

In this way, gravitational and inertial mass, as well as electric mass (charge), obtain a totally geometric interpretation in terms of the angle of a worldline through the time plane. Referring especially to (5.9), movement through ordinary time x^0 contributes to gravitational and inertial mass, movement through axial time x^5 contributes to electrical mass, and real "force" and "absolute acceleration" arises from angular movement through the time plane in which a worldline projects both x^0 and x^5 components.

6. Symmetric Gravitation and Antisymmetric Electrodynamics: Can they be Compatible?

Now, let us turn back to the association $F^{\mu}{}_{\tau} \propto 2\Gamma^{\mu}{}_{\tau 5}$ in (5.4). One of the fundamental difficulties which has been encountered by physicists attempting to unify classical electrodynamics with GTR, is that the former is an antisymmetric field theory while the latter is symmetric. How to combine "oil" and "water" in this way has perplexed physicists for over a century. So, the question arises, does the relationship $F^{\mu}{}_{\tau} \propto 2\Gamma^{\mu}{}_{\tau 5}$ provide a path for a seamless and internally-consistent union of these two theories?

As it stands, $F^{\mu}{}_{\tau}$ is a mixed tensor, and it would be better to raise this into contravariant form where we can clearly examine the consequences of having an antisymmetric field strength tensor $F^{\mu\nu} = -F^{\nu\mu}$. However, now that we are in 5-D, we have to be careful that we are raising indexes properly, using the full g^{MN} and not just its four-dimensional subset. To do this, we need to recognize that in 5-D, (5.4) should be generalized to $F^{\mathrm{M}}{}_{\mathrm{T}} \propto 2\Gamma^{\mathrm{M}}{}_{\mathrm{T}5}$, which means that there are four additional, independent components in $F^{\mathrm{M}}{}_{\mathrm{T}}$ (assuming we maintain an antisymmetric field strength by requiring that $F^{\mathrm{MN}} = -F^{\mathrm{NM}}$). We are not at this juncture concerned about what these new components might be; the only reason for using $F^{\mathrm{M}}{}_{\mathrm{T}}$ rather than $F^{\mu}{}_{\tau}$ is to make sure we handle the raising of the lower index properly in 5-dimensions.

So, going into 5-D, and using $\Gamma^{\mathrm{M}}{}_{\mathrm{T}5} = \frac{1}{2} g^{\mathrm{MA}}\left(g_{\mathrm{AT},5} + g_{5\mathrm{A},\mathrm{T}} - g_{\mathrm{T}5,\mathrm{A}}\right)$, we rewrite (5.4) as:

$$\frac{1}{2} F^{\mathrm{MN}} = \frac{1}{2} g^{\mathrm{TN}} F^{\mathrm{M}}{}_{\mathrm{T}} \propto g^{\mathrm{TN}} \Gamma^{\mathrm{M}}{}_{\mathrm{T}5} = \frac{1}{2} g^{\mathrm{TN}} g^{\mathrm{MA}}\left(g_{\mathrm{AT},5} + g_{5\mathrm{A},\mathrm{T}} - g_{\mathrm{T}5,\mathrm{A}}\right). \tag{6.1}$$

Although unconcerned for now about the extra components in F^{MN}, we shall follow the customary path and regard F^{MN} as a totally-antisymmetric tensor, thereby extending this basic property of electrodynamics to these extra components, whatever they may be. That is, we continue to employ the condition $F^{\mathrm{MN}} = -F^{\mathrm{NM}}$.

Combining $F^{\mathrm{MN}} = -F^{\mathrm{NM}}$ with (6.1) now lets us write:

$$F^{\mathrm{MN}} = -F^{\mathrm{NM}} \propto g^{\mathrm{TN}} g^{\mathrm{MA}}\left(g_{\mathrm{AT},5} + g_{5\mathrm{A},\mathrm{T}} - g_{\mathrm{T}5,\mathrm{A}}\right) = -g^{\mathrm{TM}} g^{\mathrm{NA}}\left(g_{\mathrm{AT},5} + g_{5\mathrm{A},\mathrm{T}} - g_{\mathrm{T}5,\mathrm{A}}\right). \tag{6.2}$$

This lets us express the antisymmetric field strength relation $F^{\mathrm{MN}} = -F^{\mathrm{NM}}$ completely in terms of certain relationships involving first derivatives of the gravitational potential, as expressed via the metric tensor. Now, let us reduce this.

From (6.2), we can rename indexes and use the symmetry of the metric tensor to write:

$$g^{\mathrm{TM}} g^{\mathrm{NA}}\left(g_{\mathrm{TA},5} + g_{5\mathrm{T},\mathrm{A}} - g_{\mathrm{A}5,\mathrm{T}}\right) = g^{\mathrm{TM}} g^{\mathrm{NA}}\left(-g_{\mathrm{TA},5} + g_{5\mathrm{T},\mathrm{A}} - g_{\mathrm{A}5,\mathrm{T}}\right), \tag{6.3}$$

which further reduces with some index changes and the symmetry of the metric tensor to:

$$g^{\mathrm{M}\Sigma} g^{\mathrm{TN}} g_{\Sigma\mathrm{T},5} = 0. \tag{6.4}$$

This is an alternative way of saying that $F^{\mathrm{MN}} = -F^{\mathrm{NM}}$.

We can further simplify this using the inverse relationship $g^{\mathrm{TN}} g_{\Sigma\mathrm{T}} = \delta^{\mathrm{N}}{}_{\Sigma}$, which we can differentiate with respect to the 5$^{\text{th}}$ dimension to obtain $\left(g^{\mathrm{TN}} g_{\Sigma\mathrm{T}}\right)_{,5} = g^{\mathrm{TN}}{}_{,5} g_{\Sigma\mathrm{T}} + g^{\mathrm{TN}} g_{\Sigma\mathrm{T},5} = 0$, i.e., $g^{\mathrm{TN}} g_{\Sigma\mathrm{T},5} = -g^{\mathrm{TN}}{}_{,5} g_{\Sigma\mathrm{T}}$. This can then be used to reduce (6.4) to the very simple:

$$g^{\mathrm{MN}}{}_{,5} = 0. \tag{6.5}$$

The above, $g^{\mathrm{MN}}{}_{,5}=0$, is a purely geometric statement *completely equivalent to* $F^{\mathrm{MN}}=-F^{\mathrm{NM}}$. The symmetric field theory of gravitation is fully compatible with the antisymmetric field theory of electrodynamics, *so long as we require that* $g^{\mathrm{MN}}{}_{,5}=0$.

7. Do Maxwell's Equations Become Components of Einstein's Gravitational Field Equation?

We have shown the possibility that Lorentz force motion might be described as simple geodesic motion in a five-dimensional spacetime with axial time, and that the inequality of electrical and inertial mass which causes one to "feel" a force and prevents one from falling through the floor of Einstein's elevator or into the earth's core, may well emanate from the simple proportionality $-\frac{q}{m}\propto\frac{dx^5}{d\tau}$. But equations of motion are only one part of a complete field theory. The other part is a specification of how the "sources" of that theory influence the "fields" originating from those sources. In a complete theory, the equations of motion then describe motion through the fields originating from the sources.

To complete the field theory which we have motivated thus far, one therefore would also need to also examine the Einstein equation in five dimensions:

$$-\kappa T^{\mathrm{M}}{}_{\mathrm{N}}=R^{\mathrm{M}}{}_{\mathrm{N}}-\tfrac{1}{2}\delta^{\mathrm{M}}{}_{\mathrm{N}}R\,, \qquad (7.1)$$

as well as the Riemann Identity:

$$R_{\mathrm{MNAB}}+R_{\mathrm{MABN}}+R_{\mathrm{MBNA}}=0\,, \qquad (7.2)$$

to see if among their new axial (index = 5) components, one might find the Maxwell equations $j^{\nu}=F^{\mu\nu}{}_{;\mu}$, and $F^{\mu\nu;\sigma}+F^{\nu\sigma;\mu}+F^{\sigma\mu;\nu}=0$.

In five dimensions, one would of course specify the Riemann tensor in the usual way, albeit with an extra "5" index. That is:

$$R^{\mathrm{A}}{}_{\mathrm{BMN}}=-\Gamma^{\mathrm{A}}{}_{\mathrm{BM,N}}+\Gamma^{\mathrm{A}}{}_{\mathrm{BN,M}}+\Gamma^{\Sigma}{}_{\mathrm{BN}}\Gamma^{\mathrm{A}}{}_{\Sigma\mathrm{M}}-\Gamma^{\Sigma}{}_{\mathrm{BM}}\Gamma^{\mathrm{A}}{}_{\Sigma\mathrm{N}}\,. \qquad (7.3)$$

Now, let's consider the $\mathrm{M}=5$ component of this equation, that is:

$$R^{\mathrm{A}}{}_{\mathrm{B5N}}=-\Gamma^{\mathrm{A}}{}_{\mathrm{B5,N}}+\Gamma^{\mathrm{A}}{}_{\mathrm{BN,5}}+\Gamma^{\Sigma}{}_{\mathrm{BN}}\Gamma^{\mathrm{A}}{}_{\Sigma5}-\Gamma^{\Sigma}{}_{\mathrm{B5}}\Gamma^{\mathrm{A}}{}_{\Sigma\mathrm{N}}\,. \qquad (7.4)$$

The second term in the above, with explicit substitution of $\Gamma^{\mathrm{A}}{}_{\mathrm{BN}}$ is given by:

$$\Gamma^{A}{}_{BN,5} = \tfrac{1}{2}\left[g^{A\Sigma}\left(g_{\Sigma B,N} + g_{N\Sigma,B} - g_{BN,\Sigma}\right)\right]_{,5} = 0 . \tag{7.5}$$

This is equal to zero, as a consequence of $g^{MN}{}_{,5} = 0$, equation (6.5), which is the same thing as $F^{MN} = -F^{NM}$, and because ordinary derivatives commute. Thus, by virtue of $F^{MN} = -F^{NM}$ a.k.a. $g^{MN}{}_{,5} = 0$, (7.4) simplifies to:

$$R^{A}{}_{B5N} = -\Gamma^{A}{}_{B5,N} + \Gamma^{\Sigma}{}_{BN}\Gamma^{A}{}_{\Sigma 5} - \Gamma^{\Sigma}{}_{B5}\Gamma^{A}{}_{\Sigma N} , \tag{7.6}$$

consisting of only three terms.

Now, we make use of $\Gamma^{M}{}_{T5} \propto \tfrac{1}{2} F^{M}{}_{T}$ generalized from (5.4), to rewrite (7.6) as a proportionality, in terms of the field strength tensor, as such:

$$R^{A}{}_{B5N} \propto -F^{A}{}_{B,N} + \Gamma^{\Sigma}{}_{BN} F^{A}{}_{\Sigma} - \Gamma^{A}{}_{\Sigma N} F^{\Sigma}{}_{B} = -F^{A}{}_{B;N} . \tag{7.7}$$

What is absolutely fascinating, and of enormous eventual import, is that this expression for the mixed field strength tensor $F^{A}{}_{B}$ is *identical to its gravitationally-covariant derivative* $F^{A}{}_{B;N}$. From here, we can get to both of Maxwell's equations almost immediately.

First, let's contract (7.7) down to the Ricci tensor, and use the proportionality liberally to eliminate the minus sign, as such:

$$R_{B5} = R^{A}{}_{B5A} \propto F^{A}{}_{B;A} = j_{B} , \tag{7.8}$$

where j_{B} is the five-dimensional, covariant electric source current. Because $g^{BM}{}_{;A} = 0$, this can be raised into mixed form with some index renaming as:

$$R^{M}{}_{5} \propto F^{\Sigma M}{}_{;\Sigma} = F^{\sigma M}{}_{;\sigma} + F^{5M}{}_{;5} = F^{\sigma M}{}_{;\sigma} = j^{M} . \tag{7.9}$$

Note that $F^{5M}{}_{;5} = 0$, and so $F^{\Sigma M}{}_{;\Sigma} = F^{\sigma M}{}_{;\sigma}$, because $\Gamma^{5}{}_{A5;5} \propto \tfrac{1}{2} F^{5}{}_{A;5} = 0$ by virtue of $g^{MN}{}_{,5} = 0$, that is, $F^{MN} = -F^{NM}$.

Now, we can return to Einstein's equation (7.1), for $M = \mu$ and $N = 5$, and use (7.9) as well as $\delta^{\mu}{}_{5} = 0$, to write:

$$-\kappa T^{\mu}{}_{5} = R^{\mu}{}_{5} - \tfrac{1}{2}\delta^{\mu}{}_{5} R = R^{\mu}{}_{5} \propto F^{\sigma\mu}{}_{;\sigma} = j^{\mu} , \tag{7.10}$$

This is the first of Maxwell's equations, for the field of an electric charge. We find, in particular, that $j^{\mu} \propto T^{\mu}{}_{5}$ is a four-vector situated along the axial components of the energy momentum tensor. The electric current, which is an electrical source density, is now also simply part and parcel of the generalized gravitational source $T^{\mathrm{M}}{}_{\mathrm{N}}$!

What about Maxwell's magnetic equation? Here, we lower all indexes in (7.7) and then use the symmetry $R_{\mathrm{ABMN}} = R_{\mathrm{MNAB}}$ to rewrite (7.7) as:

$$R_{5\mathrm{NAB}} \propto F_{\mathrm{AB;N}} . \tag{7.11}$$

Then, we turn to some more geometric "marble," namely the Riemann identity (7.2). Taking the $\mathrm{M}=5$ component of this identity, we write:

$$R_{5\mathrm{NAB}} + R_{5\mathrm{ABN}} + R_{5\mathrm{BNA}} \propto F_{\mathrm{AB;N}} + F_{\mathrm{BN;A}} + F_{\mathrm{NA;B}} = 0 , \tag{7.12}$$

We may then consider the spacetime subset equations, to write:

$$R_{5\nu\alpha\beta} + R_{5\alpha\beta\nu} + R_{5\beta\nu\alpha} \propto F_{\alpha\beta;\mu} + F_{\beta\mu;\alpha} + F_{\mu\alpha;\beta} = 0 , \tag{7.13}$$

Now, Maxwell's magnetic equation also rests on geometric "marble."

Maxwell's electrodynamics in this manner, becomes fully unified with Einstein's gravitation. The equation for an electric source, $F^{\sigma\mu}{}_{;\sigma} = j^{\mu}$, is specified in geometry as:

$$-\kappa T^{\mu}{}_{5} = R^{\mu}{}_{5} - \tfrac{1}{2}\delta^{\mu}{}_{5} R . \tag{7.14}$$

The magnetic equation $F_{\alpha\beta;\mu} + F_{\beta\mu;\alpha} + F_{\mu\alpha;\beta} = 0$ is specified in geometry as:

$$R_{5\nu\alpha\beta} + R_{5\alpha\beta\nu} + R_{5\beta\nu\alpha} = 0 , \tag{7.15}$$

Finally, the geodesic equation $\frac{d^2x^{\mu}}{d\tau^2} + \Gamma^{\mu}{}_{\sigma\tau}\frac{dx^{\sigma}}{d\tau}\frac{dx^{\tau}}{d\tau} - \frac{q}{m}F^{\mu}{}_{\tau}\frac{dx^{\tau}}{d\tau} = 0$, which includes the Lorentz force law, is specified in geometry as equation (4.9):

$$\frac{d^2x^{\mu}}{d\tau^2} + \Gamma^{\mu}{}_{\sigma\tau}\frac{dx^{\sigma}}{d\tau}\frac{dx^{\tau}}{d\tau} + 2\Gamma^{\mu}{}_{\tau 5}\frac{dx^{5}}{d\tau}\frac{dx^{\tau}}{d\tau} + \Gamma^{\mu}{}_{55}\frac{dx^{5}}{d\tau}\frac{dx^{5}}{d\tau} = 0 . \tag{7.16}$$

(We note, as an aside, that $F^{\mathrm{MN}} = -F^{\mathrm{NM}}$ a.k.a. $g^{\mathrm{MN}}{}_{,5} = 0$ implies that $\Gamma^{\mu}{}_{55} = -\tfrac{1}{2} g_{55}{}^{,\mu}$. Other than to match the Lorentz force law term for term, see (5.1) and (5.2), there is no apparent fundamental reason why we must have $\Gamma^{\mu}{}_{55} = 0$. If $\Gamma^{\mu}{}_{55} = -\tfrac{1}{2} g_{55}{}^{,\mu} \neq 0$, the final term in (7.16) is an extra term in the Lorentz force law, and using $-\frac{q}{m} \propto \frac{dx^5}{d\tau}$ from (5.9), this term is of the form $\Gamma^{\mu}{}_{55} \frac{dx^5}{d\tau}\frac{dx^5}{d\tau} \propto -\frac{1}{2} g_{55}{}^{,\mu} \frac{q^2}{m^2} \propto \frac{1}{2} F^{\mu}{}_5 \frac{q^2}{m^2}$, and so includes the *coupling ratio* $\frac{q^2}{m^2}$. If $\Gamma^{\mu}{}_{55} = -\frac{1}{2} g_{55}{}^{,\mu} = 0$, then this term drops out entirely, and we revert to the exact comparison made between (5.1) and (5.2). Additionally, if $g_{55}{}^{,\mu} = 0$, and given that $g^{\mathrm{MN}}{}_{,5} = 0$ so $g^{55}{}_{,5} = 0$, this means taken together that $g_{55}{}^{,\Sigma} = 0$, so that g_{55} = constant throughout the five-dimensional spacetime geometry. If we presume that the five-geometry is locally, asymptotically flat and therefore can always be transformed into "geodesic coordinates" at a single 5-dimensional event, then because $g_{55} = +1$ in geodesic coordinates (i.e., at a single "event,") it must also be +1 everywhere else. So the condition $\Gamma^{\mu}{}_{55} = -\frac{1}{2} g_{55}{}^{,\mu} = 0$ would require that $g_{55} = +1$, everywhere.)

Irrespective of the ultimate disposition of $\Gamma^{\mu}{}_{55} = -\frac{1}{2} g_{55}{}^{,\mu}$, all of the foregoing does appear to place Maxwell's electrodynamics onto the solid geometric footing of Einstein's gravitational theory. Even the inequivalence of electrical and inertial mass, and the real, measurable forces and the absolute accelerations which accompany this, are nevertheless the result of material bodies pursuing geodesic worldlines through a five-dimensional spacetime geometry.

Lab Note 2, Part 3: Gravitational and Electrodynamic Potentials, the Electro-Gravitational Lagrangian, and a Possible Approach to Quantum Gravitation

Jay R. Yablon (jyablon@nycap.rr.com), February 14, 2008

8. The Electrodynamic Potential as the Axial Component of the Gravitational Potential

Working from the relationship $F^{\mathrm{M}}{}_{\mathrm{T}} \propto 2\Gamma^{\mathrm{M}}{}_{\mathrm{T5}}$ which generalizes (5.4) to five dimensions, and recognizing that the field strength tensor $F^{\mu\nu}$ is related to the four-vector potential $A^{\mu} \equiv (\phi, A_1, A_2, A_3)$ according to $F^{\mu\nu} = A^{\mu;\nu} - A^{\nu;\mu}$, let us now examine the relationship between A^{μ} and the metric tensor g_{MN}. This is important for several reasons, one of which is that these are both fields and so should be compatible in some manner at the same differential order, and not the least of which is that the vector potential A^{μ} is necessary to establish the QED Lagrangian, and to thereby treat electromagnetism quantum-mechanically. (See, e.g., Witten, E., *Duality, Spacetime and Quantum Mechanics*, Physics Today, May 1997, pg. 28.)

Starting with $\frac{1}{2}F^{\mathrm{M}}{}_{\mathrm{T}} \propto \Gamma^{\mathrm{M}}{}_{\mathrm{T5}}$, expanding the Christoffel connections $\Gamma^{\mathrm{A}}{}_{\mathrm{BN}} = \frac{1}{2}g^{\mathrm{A}\Sigma}\left(g_{\Sigma\mathrm{B,N}} + g_{\mathrm{N}\Sigma\mathrm{,B}} - g_{\mathrm{BN},\Sigma}\right)$, making use of $g^{\mathrm{MN}}{}_{,5} = 0$ which as shown in (6.5) is equivalent to $F^{\mathrm{MN}} = -F^{\mathrm{NM}}$, and using the symmetry of the metric tensor, we may write:

$$\frac{1}{2}F^{\mathrm{M}}{}_{\mathrm{T}} \propto \Gamma^{\mathrm{M}}{}_{\mathrm{T5}} = \frac{1}{2}g^{\mathrm{M}\Sigma}\left(g_{\Sigma\mathrm{T},5} + g_{5\Sigma,\mathrm{T}} - g_{\mathrm{T5},\Sigma}\right) = \frac{1}{2}g^{\mathrm{M}\Sigma}\left(g_{5\Sigma,\mathrm{T}} - g_{5\mathrm{T},\Sigma}\right). \tag{8.1}$$

It is helpful to lower the indexes in field strength tensor and connect this to the covariant potentials A_{μ}, generalized into 5-dimensions as A_{M}, using $F_{\Sigma\mathrm{T}} \equiv A_{\Sigma;\mathrm{T}} - A_{\mathrm{T};\Sigma}$, as such:

$$A_{\Sigma;\mathrm{T}} - A_{\mathrm{T};\Sigma} \equiv F_{\Sigma\mathrm{T}} = g_{\Sigma\mathrm{M}}F^{\mathrm{M}}{}_{\mathrm{T}} \propto g_{\Sigma\mathrm{M}}g^{\mathrm{MA}}\left(g_{5\mathrm{A,T}} - g_{5\mathrm{T,A}}\right) = \left(g_{5\Sigma,\mathrm{T}} - g_{5\mathrm{T},\Sigma}\right). \tag{8.2}$$

The relationship $F_{\Sigma\mathrm{T}} \propto \left(g_{5\Sigma,\mathrm{T}} - g_{5\mathrm{T},\Sigma}\right)$ expresses clearly, the antisymmetry of $F_{\Sigma\mathrm{T}}$ in terms of the remaining connection terms involving the gravitational potential. Of particular interest, is that we may deduce from (8.2), the proportionality

$$A_{\Sigma;\mathrm{T}} \propto g_{5\Sigma,\mathrm{T}}. \tag{8.3}$$

(If one forms $A_{\Sigma;\mathrm{T}} - A_{\mathrm{T};\Sigma}$ from (8.3) and then renames indexes and uses $g_{\mathrm{MN}} = g_{\mathrm{NM}}$, one arrives back at (8.2).) Further, we well know that $F_{\Sigma\mathrm{T}} = A_{\Sigma;\mathrm{T}} - A_{\mathrm{T};\Sigma} = A_{\Sigma,\mathrm{T}} - A_{\mathrm{T},\Sigma}$, i.e., that the covariant derivatives of the potentials cancel out so as to become ordinary derivatives when specifying $F_{\Sigma \mathrm{T}}$, i.e., that $F_{\Sigma \mathrm{T}}$ is invariant under the transformation $A_{\Sigma;\mathrm{T}} \to A_{\Sigma,\mathrm{T}}$. Additionally, the Maxwell components (7.10) of the Einstein equation, are also invariant under $A_{\Sigma;\mathrm{T}} \to A_{\Sigma,\mathrm{T}}$, because (7.10) also employs only the field strength $F^{\sigma\mu}$. Therefore, let is transform $A_{\Sigma;\mathrm{T}} \to A_{\Sigma,\mathrm{T}}$ in the above, then perform an ordinary integration and index renaming, to write:

$$A_{\mathrm{M}} \propto g_{5\mathrm{M}} . \tag{8.4}$$

In the four spacetime dimensions, this means that the axial portion of the metric tensor is proportional to the vector potential, $g_{5\mu} \propto A_{\mu}$, and that the field strength tensor $F_{\Sigma \mathrm{T}}$ and the gravitational field equations $-\kappa T^{\mathrm{M}}{}_{\mathrm{N}} = R^{\mathrm{M}}{}_{\mathrm{N}} - \frac{1}{2}\delta^{\mathrm{M}}{}_{\mathrm{N}} R$ are invariant under the transformation $A_{\Sigma;\mathrm{T}} \to A_{\Sigma,\mathrm{T}}$ used to arrive at (8.4). We choose to set $A_{\Sigma;\mathrm{T}} \to A_{\Sigma,\mathrm{T}}$, and can thereby employ the integrated relationship (8.4) in lieu of the differential equation (8.3), with no impact at all on the electromagnetic field strength or the gravitational field equations, which are invariant with respect to this choice.

9. Unification of the Gravitational and QED Lagrangians

The Lagrangian density for a gravitational field *in vacuo* is $\mathcal{L}_{gravitation} = \sqrt{-g}R$, where g is the metric tensor determinant and $R = g^{\mu\nu}R_{\mu\nu}$ is the Ricci tensor. Let us now examine a Lagrangian based upon the 5-dimensional Ricci scalar, which we specify by:

$$\mathfrak{R} \equiv R^{\Sigma}{}_{\Sigma} = R^{\sigma}{}_{\sigma} + R^{5}{}_{5} = R + R^{5}{}_{5} . \tag{9.1}$$

To start, we return to deduce the $\mathrm{B} = 5$ component of (7.6), namely:

$$R^{\mathrm{A}}{}_{55\mathrm{N}} = -\Gamma^{\mathrm{A}}{}_{55,\mathrm{N}} + \Gamma^{\Sigma}{}_{5\mathrm{N}}\Gamma^{\mathrm{A}}{}_{\Sigma 5} - \Gamma^{\Sigma}{}_{55}\Gamma^{\mathrm{A}}{}_{\Sigma \mathrm{N}} , \tag{9.2}$$

as well as R_{55}, which is easily found by contracting the remaining free indexes in the above:

$$R_{55} = R^{\mathrm{T}}{}_{55\mathrm{T}} = -\Gamma^{\mathrm{T}}{}_{55,\mathrm{T}} + \Gamma^{\Sigma}{}_{5\mathrm{T}}\Gamma^{\mathrm{T}}{}_{\Sigma 5} - \Gamma^{\Sigma}{}_{55}\Gamma^{\mathrm{T}}{}_{\Sigma \mathrm{T}} . \tag{9.3}$$

Now, let us return to the discussion in the next-to-last paragraph of section 7, where we considered, but reached no conclusions about, the question of whether $\Gamma^{\mu}{}_{55} = -\frac{1}{2} g_{55}{}^{,\mu}$ is, or is not, equal to zero. Let us now make the (inductive) hypothesis that $\Gamma^{\mu}{}_{55} = -\frac{1}{2} g_{55}{}^{,\mu} = 0$, and see what (deductive) results emerge from this hypothesis.

First, as already noted, taken together with $g^{\mathrm{MN}}{}_{,5} = 0$, and because $g_{55} = +1$ in geodesic coordinates, this means that that $g_{55} = +1 = \text{constant}$ *everywhere* in the 5-dimensional spacetime. Second, the geodesic equation (4.9) now does reduce to (5.2), which, via (5.3) and (5.4), can be made *identically equivalent with the Lorentz force law* (5.1). Third, from (8.4), $A_{\mathrm{M}} \propto g_{5\mathrm{M}}$, the axial component of the covariant vector potential must now be constant everywhere in spacetime, that is $A_5 \propto g_{55} = +1 = \text{constant}$. Fourth, this means that the axial components of the field strength tensor $F_{\Sigma\mathrm{T}} = A_{\Sigma;\mathrm{T}} - A_{\mathrm{T};\Sigma} = A_{\Sigma,\mathrm{T}} - A_{\mathrm{T},\Sigma}$ must all become zero. To see this, we simply take:

$$F_{5\mathrm{T}} = A_{5;\mathrm{T}} - A_{\mathrm{T};5} = A_{5,\mathrm{T}} - A_{\mathrm{T},5} \propto g_{55,\mathrm{T}} - g_{5\mathrm{T},5} = 0 . \tag{9.4}$$

The first term, $g_{55,\mathrm{T}} = 0$, by virtue of the hypothesis just made. The latter term, $g_{5\mathrm{T},5} = 0$, because this is just a component of $g^{\mathrm{MN}}{}_{,5} = 0$, i.e., $F^{\mathrm{MN}} = -F^{\mathrm{NM}}$. Another way of stating (9.4), is that only the ordinary spacetime components $F^{\mu\nu}$ of the field strength tensor F^{MN} are non-zero. Earlier, we generalized $F^{\mu\nu}$ to F^{MN}. Now, we find that all of these added components are zero. Fifth, and finally, $\Gamma^{\mu}{}_{55} = -\frac{1}{2} g_{55}{}^{,\mu} = 0$ directly simplifies (9.2) to:

$$R^{\mathrm{A}}{}_{55\mathrm{N}} = +\Gamma^{\Sigma}{}_{5\mathrm{N}}\Gamma^{\mathrm{A}}{}_{\Sigma 5} , \tag{9.5}$$

and (9.3) to:

$$R_{55} = R^{\mathrm{T}}{}_{55\mathrm{T}} = +\Gamma^{\Sigma}{}_{5\mathrm{T}}\Gamma^{\mathrm{T}}{}_{\Sigma 5} . \tag{9.6}$$

Now, it will be helpful to start with the mixed Ricci tensor $R^{\Sigma}{}_{\mathrm{N}}$, and lower this into covariant form, in a 5-covariant manner, as such:

$$R_{\mathrm{MN}} = g_{\mathrm{M}\Sigma} R^{\Sigma}{}_{\mathrm{N}} = g_{\mathrm{M}\sigma} R^{\sigma}{}_{\mathrm{N}} + g_{\mathrm{M}5} R^{5}{}_{\mathrm{N}} . \tag{9.7}$$

From this, is it easily found, making use of $g_{55} = +1$, that the component equation:

$$R_{55} = g_{5\sigma} R^{\sigma}{}_{5} + g_{55} R^{5}{}_{5} = g_{5\sigma} R^{\sigma}{}_{5} + R^{5}{}_{5} . \tag{9.8}$$

We then rearrange this into $R^{5}{}_{5} = R_{55} - g_{5\sigma}R^{\sigma}{}_{5}$ and insert the result into (9.1), thus arriving at:

$$\Re = R + R_{55} - g_{5\sigma}R^{\sigma}{}_{5}. \tag{9.9}$$

Finally, we make use of the spacetime components of: (7.9) written as $R^{\sigma}{}_{5} \propto j^{\sigma}$; (8.4) written as $g_{5\sigma} \propto A_{\sigma}$; (9.6), written as $R_{55} = +\Gamma^{\Sigma}{}_{5\mathrm{T}}\Gamma^{\mathrm{T}}{}_{\Sigma 5}$; and the oft-employed $\Gamma^{\mathrm{M}}{}_{\mathrm{T}5} \propto \frac{1}{2}F^{\mathrm{M}}{}_{\mathrm{T}}$, to rewrite (9.9) as:

$$\Re = R + R_{55} - g_{5\sigma}R^{\sigma}{}_{5} = R + \Gamma^{\Sigma}{}_{5\mathrm{T}}\Gamma^{\mathrm{T}}{}_{\Sigma 5} - g_{5\sigma}R^{\sigma}{}_{5} = R + a\cdot\tfrac{1}{4}F^{\Sigma}{}_{\mathrm{T}}F^{\mathrm{T}}{}_{\Sigma} - b\cdot A_{\sigma}j^{\sigma}, \tag{9.10}$$

where we have absorbed the proportionality $\propto$ into the unknown constants a, b. However, the term $F^{\Sigma}{}_{\mathrm{T}}F^{\mathrm{T}}{}_{\Sigma} = F^{\Sigma\mathrm{T}}F_{\mathrm{T}\Sigma} = -F^{\Sigma\mathrm{T}}F_{\Sigma\mathrm{T}} = -F^{\sigma\tau}F_{\sigma\tau}$, with the final step taken by virtue of $F_{5\mathrm{T}} = 0$ from (9.4). Now, choosing $a = b = 1$, (9.10) finally reduces to:

$$\Re = R - \tfrac{1}{4}F^{\sigma\tau}F_{\sigma\tau} - A_{\sigma}j^{\sigma} = R + \tfrac{1}{\sqrt{-g}}\mathfrak{L}_{QED} = \tfrac{1}{\sqrt{-g}}\left(\mathfrak{L}_{gravitation} + \mathfrak{L}_{QED}\right). \tag{9.11}$$

Lo and behold: the QED Lagrangian density $\mathfrak{L}_{QED} = \sqrt{-g}\left(-\tfrac{1}{4}F^{\sigma\tau}F_{\sigma\tau} - A_{\sigma}j^{\sigma}\right)$ is *automatically* added to the four-dimensional Ricci scalar R as part of the five-dimensional Ricci scalar $\Re$. More to the point: $\sqrt{-g}\Re$ is a seamlessly-integrated electro-gravitational Lagrangian density, *in vacuo*. Choosing the constant factors $a = b = 1$, *even the factor of ¼ and the negative sign of the QED Lagrangian density are all automatically introduced.* Employed in the Euler-Lagrange equation, $\mathfrak{L}_{QED}$ can be used in the usual manner to obtain Maxwell's equation $j^{\nu} = F^{\mu\nu}{}_{;\mu}$. But of even greater interest, is that we now bring QED into the mix, directly from gravitational theory in five dimensions, which raises a possible approach to quantum gravitation.

10. A Tentative Path Toward Quantum Gravitation

Maintaining *g* to be the four-dimensional metric determinant of $g_{\mu\nu}$, the electro-gravitational Lagrangian density, *in vacuo*, is now specified by:

$$\mathfrak{L} = \sqrt{-g}\Re = \sqrt{-g}\left(R - \tfrac{1}{4}F^{\sigma\tau}F_{\sigma\tau} - A_{\sigma}j^{\sigma}\right). \tag{10.1}$$

Though derived from a 5-dimensional spacetime with axial time, and wholly-founded upon geometrodynamics in five dimensions, all that remains in (10.1) are objects specified in ordinary

4-dimensional spacetime. The Einstein-Hilbert action – *now including QED* – is then specified in the usual way by integrating over the invariant 4-volume element $dV \equiv \sqrt{-g}d^4x$:

$$S(g_{\mu\nu}, A_\sigma) = \frac{c^4}{16\pi G}\int \sqrt{-g}\Re d^4x = \frac{c^4}{16\pi G}\int \left(R - \tfrac{1}{4}F^{\sigma\tau}F_{\sigma\tau} - A_\sigma j^\sigma\right)dV \ . \qquad (10.2)$$

Expanding $F^{\sigma\tau} = A^{\sigma;\tau} - A^{\tau;\sigma}$, and integrating by parts in the usual way, then allows one to specify a path integral $Z = \int DAe^{iS(A)} \equiv e^{iW(J)}$ and associated transition amplitude $W(J)$ for the $\mathfrak{L}_{QED}$ portion of the above. But what is particularly intriguing, is that $\frac{1}{\sqrt{-g}}\mathfrak{L}_{QED}$, in five dimensions, is naturally added to the Ricci scalar R, see (9.11). What Zee, A. in *Quantum Field Theory in a Nutshell*, Princeton (2003), pp. 167 and 460 refers to as "The Central Identity of Quantum Field Theory," is given generally by:

$$\int D\varphi e^{-\frac{1}{2}\varphi\cdot K\cdot\varphi + J\cdot\varphi - V(\varphi)} = e^{-V(\delta/\delta J)}e^{\frac{1}{2}J\cdot K^{-1}\cdot J} \ . \qquad (10.3)$$

In (10.2), we find that the Ricci curvature scalar R, in this identity, plays the role of V.

Given that the Ricci scalar R, which is a classical gravitational scalar, is firmly-entrenched together with $\mathfrak{L}_{QED}$ in equation (9.11), given that this originates strictly on the geometrodynamic basis of the 5-dimensional spacetime geometry with axial time, given that R appears to map neatly to V in the Gaussian identity (10.3), and given that we know a great deal about how $\mathfrak{L}_{QED}$ is utilized in quantum field theory, all of the foregoing may provide a new pathway for understanding how to quantize gravitation.

11. The Geometric Maxwell Tensor

Before concluding this lab note, it is also helpful to recast the Maxwell energy tensor $T^\mu{}_{\nu\,Maxwell} = -\frac{1}{4\pi}\left(F^{\mu\sigma}F_{\nu\sigma} - \frac{1}{4}\delta^\mu{}_\nu F^{\sigma\tau}F_{\sigma\tau}\right)$ into geometric form. Again we start with $\Gamma^{\mathrm{M}}{}_{\mathrm{T}5} \propto \frac{1}{2}F^{\mathrm{M}}{}_{\mathrm{T}}$, which, in light of (9.4), $F_{5\mathrm{T}} = 0$, can be written without any information loss as the four-dimensional $\Gamma^\mu{}_{\tau 5} \propto \frac{1}{2}F^\mu{}_\tau$.

First, we return to (9.5). Using $\Gamma^{\mathrm{M}}{}_{\mathrm{T}5} \propto \frac{1}{2}F^{\mathrm{M}}{}_{\mathrm{T}}$, this becomes:

$$R^{\mathrm{M}}{}_{55\mathrm{N}} = +\Gamma^{\Sigma}{}_{5\mathrm{N}}\Gamma^{\mathrm{M}}{}_{\Sigma 5} \propto -\tfrac{1}{4}F^{\mathrm{M}\Sigma}F_{\mathrm{N}\Sigma}, \qquad (11.1)$$

and with $F_{5\mathrm{T}} = 0$,

$$R^{\mu}{}_{55\nu} \propto -\tfrac{1}{4} F^{\mu\sigma} F_{\nu\sigma} . \tag{11.2}$$

The contraction of this, also evident via (9.6), becomes:

$$R_{55} \propto -\tfrac{1}{4} F^{\sigma\tau} F_{\sigma\tau} . \tag{11.3}$$

Employing (11.2) and (11.3) then enables us to specify the Maxwell tensor, geometrically, as:

$$T^{\mu}{}_{55\nu} \equiv T^{\mu}{}_{\nu\,Maxwell} = -\tfrac{1}{4\pi}\left(F^{\mu\sigma} F_{\nu\sigma} - \tfrac{1}{4}\delta^{\mu}{}_{\nu} F^{\sigma\tau} F_{\sigma\tau}\right) \propto R^{\mu}{}_{55\nu} - \tfrac{1}{4}\delta^{\mu}{}_{\nu} R_{55} . \tag{11.4}$$

To maintain a balanced set of spacetime indexes, (11.4) suggests that the Maxwell tensor must actually be the axial $\mathrm{AB} = 55$ and $\mathrm{MN} = \mu\nu$ component of larger, fourth-rank energy tensor $T^{\mathrm{M}}{}_{\mathrm{ABN}}$ which has the same symmetries as the Riemann tensor $R^{\mathrm{M}}{}_{\mathrm{ABN}}$, and for which $T^{\mu}{}_{55\nu} \equiv T^{\mu}{}_{\nu\,Maxwell}$. The Poynting vector is then to be found residing on the $T^{0}{}_{55k} \propto R^{0}{}_{55k}$ components of the Riemann tensor, $k = 1,2,3$, and so too, acquires a totally geometric foundation.

These expanded tensors have 50 independent components. Particularly, $R_{\mathrm{MABN}} = -R_{\mathrm{MANB}}$ and $R_{\mathrm{MABN}} = R_{\mathrm{NBMA}}$ yields a symmetric tensor of two antisymmetric 5x5 tensors. An antisymmetric 5x5 tensor has 10 independent components, yielding a symmetric 10x10 tensor with 55 independent components. However, identity (7.2) imposes *five* constraints among these 55 independent components, one for each of the five indexes which is *omitted* from any particular equation based on (7.2). The result is 50 independent components. Maxwell's tensor (11.4) clearly contains 10 of these independent components, leaving the remaining 40 components for further exploration.