Kaluza-Klein Theory and Lorentz Force Geodesics

DRAFT – FEBRUARY 19, 2008

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Abstract:

We examine a general Kaluza-Klein theory of classical electrodynamics and gravitation in a five-dimensional Riemannian geometry. Based solely on the condition that the electrodynamic Lorentz force law must describe geodesic motion in this five-dimensional geometry, it appears possible to place all of Maxwell's electrodynamics, the theory of electrodynamic potentials, and the QED action on a solid geometrodynamic footing, for weak and strong electro-gravitational fields. We make no choice as between the fifth dimension being timelike or spacelike, but simply point out the impact in those places where this choice makes a difference. We also show, if the fifth dimension is chosen to be a compact, cylindrical spacelike dimension, that motion in this fifth dimension may be synonymous with intrinsic spin, and that the radius of the compact dimension, for a electromagnetic coupling on the order of unity, is equal to the Schwarzschild radius of the geometrodynamic vacuum first explored by Wheeler.

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1. Introduction

The possibility of employing a fifth spacetime dimension to unite classical gravitation and electrodynamics has intrigued physicists for almost a century. [1], [2] Early theorists became perhaps overly-occupied with making assumptions about the scale or topology of the extra coordinate dimension. [3] Following the path of Wesson and other current-day theorists [4], we seek here to expose the main features of Kaluza- Klein theory irrespective of any particular model, and most importantly, to make the connection between Einstein's gravitation and Maxwell's electrodynamics which is offered by Kaluza-Klein theories as clear and solid as possible, and as independent as possible of the detailed choice of model.

Most fundamentally, we adopt the view of the above-noted theorists that matter and electrodynamic charge are "induced" in observed four dimensions of spacetime, from a vacuum in five dimensions, and so, in keeping with the spirit of Wheeler's program, [5] are of completely *geometrodynamic* origin. Particularly, we seek to show how classical electrodynamics emerges entirely from an Einstein-Hilbert Action of the general form $S = \frac{1}{k}\int R dV$ where R is a suitably-defined Ricci curvature scalar, integrated over a suitable multidimensional spacetime volume, and k is a constant. The reader will observe that this omits any Lagrangian density $\mathfrak{L}_{\text{Matter}}$ of matter, i.e., that it is *not* of the form $S = \int (kR + L_{\text{Matter}}) dV$ and so is in the nature of a vacuum action equation. In different terms, we seek to induce the entirely of Maxwell's electrodynamics with sources, particularly its Lagrangian density $\mathfrak{L}_{QED} = \left(F^{\sigma\tau} F_{\sigma\tau} - A_\mu j^\mu\right)$, $\hbar = c = 1$ out of a gravitationally-based vacuum.

The main line of development will be based on a single proposition: we shall require that *the Lorentz force of electrodynamics,* $m\dfrac{d^2 x^\mu}{d\tau^2} = qF^\mu{}_\tau \dfrac{dx^\tau}{d\tau}$, *must be represented as fully geodesic motion in the five-dimensional geometry*.

In five dimensions, we shall employ $g_{\text{MN}} \equiv g_{\text{NM}}$ with $\text{M},\text{N} = 0,1,2,3,5$ for the metric tensor, so $g_{\mu\nu}$ with $\mu,\nu = 0,1,2,3$ is the ordinary metric tensor in the spacetime subspace.

Inverses are defined in the usual manner according to $g^{\mathrm{M\Sigma}} g_{\Sigma\mathrm{N}} = \delta^{\mathrm{M}}{}_{\mathrm{N}}$ and so $g^{\mathrm{M\Sigma}}$ and $g_{\Sigma\mathrm{N}}$ raise and lower indexes in the customary manner.

While most authors treat the fifth dimension as spacelike and a few have considered this to be timelike, e.g., [6], [7], [8], we shall approach the fifth dimension as independently of this choice as possible. Where this choice does make a difference, we shall point this out. If we define $g_{\mathrm{MN}} \equiv \eta_{\mathrm{MN}} + \bar{\kappa} h_{\mathrm{MN}}$ in the usual manner with $\bar{\kappa} = \sqrt{16\pi G/\hbar c^5}$, then for the weak-field limit $g_{\mathrm{MN}} \to \eta_{\mathrm{MN}}$. If the fifth dimension is timelike, $\mathrm{diag}(\eta_{\mathrm{MN}}) = (+1,-1,-1,-1,+1)$; if it is spacelike, then $\mathrm{diag}(\eta_{\mathrm{MN}}) = (+1,-1,-1,-1,-1)$. In either case, $\eta_{\mathrm{MN}} = 0$ for $\mathrm{M} \neq \mathrm{N}$. Note that the constant κ in Einstein's equation $-\kappa T^{\mu}{}_{\nu} = R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu} R$ is related to the foregoing $\bar{\kappa}$, with fundamental constants restored, by $\kappa = \frac{1}{2}\hbar c \bar{\kappa}^2 = 8\pi G/c^4$, with the overbar used to distinguish these two constants $\kappa, \bar{\kappa}$.

2. Geodesic Motion in Five Dimensions, and the Lorentz Force

We start by maintaining the usual interval in the 4-dimensional spacetime subspace, using $d\tau^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}$, and define the five-space interval as:

$$\begin{aligned} d\mathrm{T}^2 \equiv g_{\mathrm{MN}} dx^{\mathrm{M}} dx^{\mathrm{N}} &= g_{\mu\nu} dx^{\mu} dx^{\nu} + g_{5\nu} dx^5 dx^{\nu} + g_{\mu 5} dx^{\mu} dx^5 + g_{55} dx^5 dx^5 \\ &= d\tau^2 + 2 g_{5\sigma} dx^5 dx^{\sigma} + g_{55} dx^5 dx^5 \end{aligned} . \tag{2.1}$$

The above is independent of whether the weak field $g_{55} \to \eta_{55} = \pm 1$, i.e., of whether the fifth dimension is timelike or spacelike, and is generally model-independent.

Like any metric equation, (2.1) can be algebraically-manipulated into:

$$1 = g_{\mathrm{MN}} \frac{dx^{\mathrm{M}}}{d\mathrm{T}} \frac{dx^{\mathrm{N}}}{d\mathrm{T}}, \tag{2.2}$$

which is the first integral of the equation of motion. In five dimensions, we specify the Christoffel connections in the usual manner, that is, $\Gamma^{\mathrm{M}}{}_{\Sigma\mathrm{T}} = \frac{1}{2} g^{\mathrm{MA}} \left(g_{\mathrm{A\Sigma,T}} + g_{\mathrm{TA,\Sigma}} - g_{\mathrm{\Sigma T,A}} \right)$, hence $\Gamma^{\mathrm{M}}{}_{\Sigma\mathrm{T}} = \Gamma^{\mathrm{M}}{}_{\mathrm{T}\Sigma}$. We employ $g_{\mathrm{MN};\Sigma} = 0$ as usual, with the usual first rank covariant derivative $A^{\mathrm{M}}{}_{;\Sigma} = A^{\mathrm{M}}{}_{,\Sigma} + \Gamma^{\mathrm{M}}{}_{\mathrm{A}\Sigma} A^{\mathrm{A}}$. We then take the covariant derivative of each side of (2.2) above, and after the usual reductions employed in four dimensions, and multiplying the result through by $d\mathrm{T}^2 / d\tau^2$, we arrive at the five-dimensional geodesic equation:

$$\frac{d^2 x^{\mathrm{M}}}{d\tau^2} + \Gamma^{\mathrm{M}}{}_{\Sigma\mathrm{T}} \frac{dx^{\Sigma}}{d\tau} \frac{dx^{\mathrm{T}}}{d\tau} = 0 . \tag{2.3}$$

The above is five independent equations. We are interested for now in the four equations for which $\mathrm{M} = \mu$, which specify motion in ordinary spacetime:

$$\frac{d^2 x^{\mu}}{d\tau^2} + \Gamma^{\mu}{}_{\Sigma\mathrm{T}} \frac{dx^{\Sigma}}{d\tau} \frac{dx^{\mathrm{T}}}{d\tau} = 0 . \tag{2.4}$$

This expands, using the metric tensor symmetry $g_{\mathrm{MN}} = g_{\mathrm{NM}}$, to:

$$\frac{d^2 x^{\mu}}{d\tau^2} + \Gamma^{\mu}{}_{\sigma\tau} \frac{dx^{\sigma}}{d\tau} \frac{dx^{\tau}}{d\tau} + 2\Gamma^{\mu}{}_{5\sigma} \frac{dx^{5}}{d\tau} \frac{dx^{\sigma}}{d\tau} + \Gamma^{\mu}{}_{55} \frac{dx^{5}}{d\tau} \frac{dx^{5}}{d\tau} = 0 . \tag{2.5}$$

Now, let us contrast (2.5) above to the gravitational geodesic equation which includes the Lorentz force law, namely, equation (20.41) of [9]:

$$\frac{d^2 x^{\mu}}{d\tau^2} + \Gamma^{\mu}{}_{\sigma\tau} \frac{dx^{\sigma}}{d\tau} \frac{dx^{\tau}}{d\tau} - \frac{q}{m} F^{\mu}{}_{\sigma} \frac{dx^{\sigma}}{d\tau} = 0 . \tag{2.6}$$

We now take a critical step: *We require that the Lorentz force as expressed above, must be represented as nothing other than geodesic motion in the five-dimensional geometry*. The first two terms in (2.5) and (2.6) are identical, and they specify geodesic motion in an ordinary gravitational field absent any electrodynamic fields or sources. The absence of any mass or charge in the first two terms captures the Galilean principle of equivalence and expresses Newtonian inertial motion in a gravitational field via the Christoffel connections $\Gamma^{\mu}{}_{\sigma\tau}$.

If we require the Lorentz force to also be fashioned as geodesic motion through geometry, then we can do so by defining the third terms in (2.5) and (2.6) to be equivalent to one another, and the fourth term in (2.5) to be zero. Therefore, we now define:

$$2\Gamma^{\mu}{}_{5\sigma} \frac{dx^{5}}{d\tau} \frac{dx^{\sigma}}{d\tau} \equiv -\frac{q}{m} F^{\mu}{}_{\sigma} \frac{dx^{\sigma}}{d\tau} \text{, and} \tag{2.7}$$

$$\Gamma^{\mu}{}_{55} \equiv 0 . \tag{2.8}$$

One might wish to consider $\Gamma^{\mu}{}_{55} \neq 0$, in which case $\Gamma^{\mu}{}_{55} \frac{dx^5}{d\tau} \frac{dx^5}{d\tau}$ in (2.5) would become an additional term in the Lorentz force law, but in the absence of experimental evidence for any deviations from the Lorentz force law, we shall proceed on the basis of (2.8).

The relationships (2.7) and (2.8), ensure that Lorentz force motion is in fact, no more and no less than geodesic motion in five dimensions. All else will be deduced from (2.7) and (2.8).

3. Placing the Lorentz Force on a Geometrodynamic Footing as Geodesic Motion

Now, let us focus on equation (2.7). We can divide out $dx^{\sigma}/d\tau$ from (2.7), and then write the remaining terms as.

$$2\Gamma^{\mu}{}_{5\sigma} \frac{dx^5}{d\tau} \equiv -\sqrt{\frac{1}{\hbar c^5}} F^{\mu}{}_{\sigma} \frac{q}{m}, \tag{3.1}$$

where we have explicitly restored $\hbar = c = 1$. Now, we separate the proportionalities $dx^5/d\tau \propto q/m$ and $2\Gamma^{\mu}{}_{5\sigma} \propto -F^{\mu}{}_{\sigma}$, and turn the proportionalities $\propto$ into equalities by restoring their dimensional and numeric constants, starting with the former proportionality.

Irrespective of whether the fifth dimension is timelike or spacelike, we take dx^5 to be given in dimensions of time, so that $dx^5/d\tau$ is a dimensionless ratio. In the event that the fifth dimension is spacelike, one need merely divide through by c. In rationalized Heaviside-Lorentz units, the electric charge strength q (for a unit charge such as the electron, muon and tauon) is related to the dimensionless (running) coupling $\alpha = q^2/4\pi\hbar c$ which approaches $\alpha \to 1/137.036$ at low energy. The value of α is the same in all systems of units but the numerical value of q is different, so it is imperative that the exact expression for $dx^5/d\tau \propto q/m$ be based on α rather than q, and be independent of where the 4π factor appears. Further, to match dimensions with $\sqrt{\hbar c}$ the mass m needs to be multiplied by a factor of $\sqrt{G}$. Taking all of this into account, we now define:

$$\frac{dx^5}{d\tau} \equiv -\frac{1}{4} \frac{\sqrt{\hbar c \alpha}}{\sqrt{G} m} = -\frac{1}{4} \frac{1}{\sqrt{4\pi G}} \frac{q}{m} = -\frac{1}{\sqrt{\hbar c^5}} \frac{1}{2\kappa} \frac{q}{m}. \tag{3.2}$$

The equivalence between the first two terms is independent of the system of units but the final term is in Heaviside-Lorentz units. There is freedom in the overall multiplicative numeric constant, which we choose to be $-\frac{1}{4}$. This choice is made because in the downstream development several sections hence, it leads to the correct constant factors in the QED Lagrangian and in the Maxwell stress energy tensor.

Then, we substitute (3.2) into (3.1) to obtain:

$$\Gamma^{\mu}{}_{5\sigma} \equiv \sqrt{\frac{16\pi G}{\hbar c^5}} F^{\mu}{}_{\sigma} = \overline{\kappa} F^{\mu}{}_{\sigma} . \tag{3.3}$$

As between (3.2) and (3.3), the placing of $-\frac{1}{4}$ in (3.3) causes $F^{\mu}{}_{\sigma}$ to be related to $\Gamma^{\mu}{}_{5\sigma}$ by the simple constant of proportionality $\overline{\kappa}$ from $g_{\mathrm{MN}} \equiv \eta_{\mathrm{MN}} + \overline{\kappa} h_{\mathrm{MN}}$. The definitions (3.2) and (3.3), together with $\Gamma^{\mu}{}_{55} \equiv 0$ from (2.8), when substituted into (2.5), turn the five-dimensional geodesic equation (2.5) into the Lorentz force law, and places this electrodynamic motion onto a totally-geometrodynamic footing. Of course, (3.3) is of further value, because it also relates the mixed field strength tensor $F^{\mu}{}_{\sigma}$ to the axial connection components $\Gamma^{\mu}{}_{5\sigma}$, and this will lead to numerous other results. Although the $\Gamma^{\mathrm{M}}{}_{\Sigma\mathrm{T}}$ not themselves tensors in general, (3.3) does suggest that that particular components $\Gamma^{\mu}{}_{5\sigma}$ do transform in the same way as the mixed tensor $F^{\mu}{}_{\sigma}$, multiplied by a the constant factor $\overline{\kappa}$.

4. Timelike versus Spacelike for the Fifth Dimension, and a Possible Connection to Intrinsic Spin

The results above are independent of whether the extra dimension is timelike or spacelike. Transforming into an "at rest" frame, $dx^1 = dx^2 = dx^3 = 0$, the spacetime metric equation $d\tau^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}$ reduces to $d\tau = \pm\sqrt{g_{00}}\, dx^0$, and (3.2) becomes:

$$\frac{dx^5}{dx^0} = \pm \frac{1}{4} \sqrt{\frac{g_{00}}{4\pi G}} \frac{q}{m} . \tag{4.1}$$

For a *timelike* fifth dimension, x^5 may be drawn as an "axial time" axis orthogonal to x^0, and the physics ratio q/m (which, by the way, results in the q/m material body in an electromagnetic field actually "feeling" a Newtonian force in the sense of $F = ma$) measures the "angle" at which the material body moves through the x^5, x^0 "time plane."

For a *spacelike* fifth dimension, where a compactified, hyper-cylindrical $x^5 \equiv R\phi$ (see [10], Figure 1) and R is a constant radius (distinguish from the Ricci scalar by context), $dx^5 \equiv R d\phi$. Substituting this into (3.2), leaving in the $\pm$ ratio obtained in (4.1), and inserting c into the first term to maintain a dimensionless equation, then yields:

$$\frac{R d\phi}{c d\tau} = \pm\frac{1}{4}\frac{\sqrt{\hbar c \alpha}}{\sqrt{G} m} = \pm\frac{1}{4}\frac{1}{\sqrt{4\pi G}}\frac{q}{m}. \tag{4.2}$$

We see that here, the physics ratio q/m measures an "angular frequency" of fifth-dimensional rotation. Interestingly, *this frequency runs inversely to the mass*, and by classical principles, this means that the angular momentum is independent of the mass, i.e., constant. If one doubles the mass, one halves the tangential velocity, while the radius stays constant. Together with the $\pm$ factor, one might suspect that this constant angular momentum is related to intrinsic spin. In fact, following this hunch, one can arrive at an exact expression for the compactification radius R, in the following manner:

Assume that x^5 is spacelike, casting one's lot with the preponderance of those who study Kaluza-Klein theory. In (4.2), move the c away from the first term and move the m over to the first term. Then, multiply all terms by another R. Everything is now dimensioned as an angular momentum, which we have just ascertained is constant irrespective of mass. So, set this all to $\pm\frac{1}{2}n\hbar$, which for $n = 1$, represents intrinsic spin. The result is as follows:

$$m\frac{R d\phi}{d\tau}R = \pm\frac{1}{4}\frac{\sqrt{\hbar c^3 \alpha}}{\sqrt{G}}R = \pm\frac{1}{4}\frac{c}{\sqrt{4\pi G}}qR = \pm\frac{1}{2}n\hbar. \tag{4.3}$$

Now, take the second and fourth terms, and solve for R with $n = 1$, to yield:

$$R = \frac{2}{\sqrt{\alpha}}\sqrt{\frac{G\hbar}{c^3}} = \frac{2}{\sqrt{\alpha}}L_P, \tag{4.4}$$

where $L_P = \sqrt{G\hbar/c^3}$ is the Planck length. *This gives a definitive size for the compactification radius, and it is very close to the Planck length.* What is of interest, is that α is a *running* coupling. At low probe energies, where $\alpha \to 1/137.036$, $R = 23.412 \cdot L_P$. However, this is just the *apparent* radius relative to the low probe energy. If one were to probe to a regime where α becomes large, say, of order unity, $\alpha = 1$ then $R = 2L_P$ is actually identical with the Schwarzschild black hole radius $R_S = 2L_P$ of the geometrodynamic vacuum "foam." [9] at §43.4, [11]* Since we have based the foregoing on a unit charge with spin ½, and since this is independent of the mass, the foregoing would appear to characterize the compactification radius R for all of the charged leptons, and to provide a geometric foundation for intrinsic spin! Further, it suggests that for $\alpha = 1$, the Schwarzschild radius of the vacuum is synonymous with the compactication radius of the fifth dimension, $R = R_S = 2L_P$. This, by the way, is another consequence of placing the $-\frac{1}{4}$ factor in (3.2).

5. Symmetric Gravitation and Antisymmetric Electrodynamics

Now, let us turn back to the association $\Gamma^{\mu}{}_{5\sigma} = \bar{\kappa} F^{\mu}{}_{\sigma}$ in (3.3), which arises from the requirement that the Lorentz force be represented as geodesic motion in five dimensions. We know that $F^{\mu\nu} = -F^{\nu\mu}$ is an antisymmetric tensor. By virtue of (3.3), this will place certain constraints on the five-dimensional Christoffel connections $\Gamma^{\mathrm{M}}{}_{\Sigma\mathrm{T}} = \frac{1}{2} g^{\mathrm{MA}} \left(g_{\mathrm{A\Sigma,T}} + g_{\mathrm{TA,\Sigma}} - g_{\mathrm{\Sigma T,A}} \right)$, and it is important to find out what these are. These constraints, in the next section, will provide the basis for placing Maxwell's equations onto a purely geometrodynamic footing.

First, because we are working in five dimensions, we will find it desirable to generalize $F^{\mu\nu}$ to F^{MN}. We make no *a priori* supposition about the additional components in F^{MN}, other than to require that they be antisymmetric, $F^{\mathrm{MN}} \equiv -F^{\mathrm{NM}}$. *Any other information about these new components is to be deduced, not imposed.* Second, we generalize (3.3) into the full five dimensions, thus:

$$\Gamma^{\mathrm{M}}{}_{5\Sigma} = \bar{\kappa} F^{\mathrm{M}}{}_{\Sigma} . \tag{5.1}$$

By virtue of (2.8), $\Gamma^{\mu}{}_{55} \equiv 0$, we may immediately deduce that:

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* By way of review, the Planck mass, defined from the term atop Newton's law as a mass for which $GM_P{}^2 = \hbar c$, is thus $M_P = \sqrt{\hbar c / G}$. In the geometrodynamic vacuum, the negative gravitational energy between Planck masses separated by the Planck length $L_P = \sqrt{G\hbar / c^3}$ precisely counterbalances and cancels the positive energy of the Planck masses themselves. The Schwarzschild radius of a Plank mass $R_S = 2GM_P / c^2 = 2\sqrt{G\hbar / c^3} = 2L_P$.

$$\Gamma^{\mu}{}_{55} = \overline{\kappa} F^{\mu}{}_{5} = 0 . \tag{5.2}$$

As it stands, $F^{\mathrm{M}}{}_{\Sigma}$ is a mixed tensor, and it would be better to raise this into contravariant form where we can clearly examine the consequences of having an antisymmetric field strength $F^{\mathrm{MN}} \equiv -F^{\mathrm{NM}}$. Thus, let us now raise the lower index in (5.1), and at the same time equate this to the Christoffel connections, as such:

$$\overline{\kappa} F^{\mathrm{MN}} = \overline{\kappa} g^{\Sigma\mathrm{N}} F^{\mathrm{M}}{}_{\Sigma} = g^{\Sigma\mathrm{N}} \Gamma^{\mathrm{M}}{}_{5\Sigma} = \tfrac{1}{2} g^{\mathrm{MA}} g^{\Sigma\mathrm{N}} \left(g_{\mathrm{A}5,\Sigma} + g_{\Sigma\mathrm{A},5} - g_{5\Sigma,\mathrm{A}} \right). \tag{5.3}$$

Now, we use (5.3) to write $F^{\mathrm{MN}} = -F^{\mathrm{NM}}$ completely in terms of the metric tensor g_{MN} and its first derivatives, as:

$$\overline{\kappa} F^{\mathrm{MN}} = -\overline{\kappa} F^{\mathrm{NM}} = g^{\mathrm{MA}} g^{\Sigma\mathrm{N}} \left(g_{\mathrm{A}5,\Sigma} + g_{\Sigma\mathrm{A},5} - g_{5\Sigma,\mathrm{A}} \right) = -g^{\mathrm{NA}} g^{\Sigma\mathrm{M}} \left(g_{\mathrm{A}5,\Sigma} + g_{\Sigma\mathrm{A},5} - g_{5\Sigma,\mathrm{A}} \right). \tag{5.4}$$

Renaming indexes, and using the symmetry of the metric tensor, this is readily reduced to::

$$g^{\mathrm{M}\Sigma} g^{\mathrm{TN}} g_{\mathrm{T}\Sigma,5} = 0 . \tag{5.5}$$

This is an alternative, geometric way of saying that $F^{\mathrm{MN}} = -F^{\mathrm{NM}}$.

We can further simplify this using the inverse relationship $g^{\mathrm{TN}} g_{\mathrm{T}\Sigma} = \delta^{\mathrm{N}}{}_{\Sigma}$, which we can differentiate to obtain $\left(g^{\mathrm{TN}} g_{\mathrm{T}\Sigma} \right)_{,\mathrm{A}} = g^{\mathrm{TN}}{}_{,\mathrm{A}} g_{\mathrm{T}\Sigma} + g^{\mathrm{TN}} g_{\mathrm{T}\Sigma,\mathrm{A}} = 0$, i.e., $g^{\mathrm{TN}} g_{\mathrm{T}\Sigma,\mathrm{A}} = -g^{\mathrm{TN}}{}_{,\mathrm{A}} g_{\mathrm{T}\Sigma}$. This can then be used with $\mathrm{A} = 5$ to reduce (5.4) to the very simple expressions, for both the covariant and contravariant metric tensor:

$$g^{\mathrm{MN}}{}_{,5} = 0 ; \; g_{\mathrm{MN},5} = 0 . \tag{5.6}$$

All components of the metric tensor are constant over the variations taking place only through the fifth dimension.

Now, we return to write out $\Gamma^{\mu}{}_{55} = \tfrac{1}{2} g^{\mu\mathrm{A}} \left(g_{\mathrm{A}5,5} + g_{5\mathrm{A},5} - g_{55,\mathrm{A}} \right) = 0$ from (2.8), see also (5.2). Combined with $g_{\mathrm{MN},5} = 0$ above and $g^{\mathrm{TN}} g_{\mathrm{T}\Sigma,\mathrm{A}} = -g^{\mathrm{TN}}{}_{,\mathrm{A}} g_{\mathrm{T}\Sigma}$ we further deduce that:

$$g^{55}{}_{,\mathrm{A}} = 0 ; \; g_{55,\mathrm{A}} = 0 \tag{5.7}$$

This means, quite importantly, that $g_{55} = \text{constant}$ and $g^{55} = \text{constant}$, everywhere in the five-dimensional geometry.

To fix these constant values, consider the weak-field limit $g_{\mathrm{MN}} \to \eta_{\mathrm{MN}}$. If the fifth dimension is timelike, $\mathrm{diag}(\eta_{\mu\nu}) = (+1,-1,-1,-1,+1)$ and $g_{55} = g^{55} = +1$. If it is spacelike (leading to the intrinsic spin results of section 4), then $\mathrm{diag}(\eta_{\mathrm{MN}}) = (+1,-1,-1,-1,-1)$ and $g_{55} = g^{55} = -1$. But, by (5.7), if the above expressions for g_{55} and g^{55} are true *anywhere*, then $g_{55} = g^{55} = +1$ or $g_{55} = g^{55} = -1$ are true *everywhere*, respectively, for a timelike or spacelike fifth dimension. In either case, timelike or spacelike, $g^{55} g_{55} = 1$. As a result, the inverse relation $g^{\mathrm{T}5} g_{\mathrm{T}5} = g^{\tau 5} g_{\tau 5} + g^{55} g_{55} = g^{\tau 5} g_{\tau 5} + 1 = \delta^5{}_5 = 1$, leads also to the null condition:

$$g^{\tau 5} g_{\tau 5} = 0 , \tag{5.8}$$

which applies *irrespective* of the timelike versus spacelike choice.

Finally, using (5.1) together with (5.6) and (5.7), we may deduce:

$$\overline{\kappa} F^5{}_5 = \Gamma^5{}_{55} = \tfrac{1}{2} g^{5\mathrm{A}} \left(g_{\mathrm{A}5,5} + g_{5\mathrm{A},5} - g_{55,\mathrm{A}} \right) = 0 . \tag{5.9}$$

Taking this together with (5.2), $\Gamma^\mu{}_{55} = \overline{\kappa} F^\mu{}_5 = 0$, we have now deduced that all of the newly-introduced axial components for the mixed field tensor are zero, i.e.,

$$\overline{\kappa} F^{\mathrm{M}}{}_5 = \Gamma^{\mathrm{M}}{}_{55} = 0 . \tag{5.10}$$

The free index above can easily be lowered to also find that the covariant:

$$F_{\mathrm{M}5} = -F_{5\mathrm{M}} = 0 . \tag{5.11}$$

But, since the non-diagonal components of $F^\mu{}_\nu$ are non-zero, one should take care to ensure that the contravariant tensor components $F^{\mathrm{M}5} = -F^{5\mathrm{M}} = 0$ as well, that is, we want to make sure that the fixed index 5 in (5.10) can properly be raised. One can employ (5.1) together with the explicit components for $\Gamma^{\mathrm{M}}{}_{5\Sigma}$ to write:

$$F^{\mathrm{MN}} = g^{\Sigma\mathrm{N}} F^{\mathrm{M}}{}_\Sigma = g^{\Sigma\mathrm{N}} \Gamma^{\mathrm{M}}{}_{5\Sigma} = \tfrac{1}{2} g^{\Sigma\mathrm{N}} g^{\mathrm{MA}} \left(g_{\mathrm{A}5,\Sigma} + g_{\Sigma\mathrm{A},5} - g_{5\Sigma,\mathrm{A}} \right). \tag{5.12}$$

Expanding this to separate the μ from the 5 components, and applying (5.6), (5.7) and (5.8) as needed, together with $F^{\mathrm{MN}} = -F^{\mathrm{NM}}$ to eliminate the only term which (5.6), (5.7) and (5.8) cannot directly eliminate, one can indeed deduce that in addition to (5.10) and (5.11):

$$F^{\mathrm{M5}} = -F^{\mathrm{5M}} = 0\,. \tag{5.13}$$

Now, the free index can be easily lowered, referring also to (5.1), to find that:

$$\overline{\kappa} F^{5}{}_{\mathrm{M}} = \Gamma^{5}{}_{5\mathrm{M}} = \Gamma^{5}{}_{\mathrm{M}5} = 0\,. \tag{5.14}$$

So, we find that all of the newly-introduced axial components of the field strength tensor, whether in raised, lowered, or mixed form, are equal to zero. Equations (5.10), $\Gamma^{\mathrm{M}}{}_{55} = 0$, and (5.14), $\Gamma^{5}{}_{5\mathrm{M}} = \Gamma^{5}{}_{\mathrm{M}5} = 0$, taken together, tell us that as well that any Christoffel connection with two or more axial indexes, is also equal to zero.

Combining (5.1) with $F^{\mathrm{M5}} = -F^{5M} = 0$ as well as $F_{\mathrm{M5}} = -F_{\mathrm{5M}} = 0$, we may deduce two further relationships:

$$g^{\Sigma\mathrm{M}}\Gamma^{5}{}_{5\Sigma} = -g^{\Sigma 5}\Gamma^{\mathrm{M}}{}_{5\Sigma} = 0 \;\text{ and }\; g_{\mathrm{TM}}\Gamma^{\mathrm{M}}{}_{55} = -g_{5\mathrm{M}}\Gamma^{\mathrm{M}}{}_{5\mathrm{T}} = 0\,, \tag{5.15}$$

which are variations of the “two or more axial index” rule noted above.

It is also helpful as we shall soon see when we examine the Riemann tensor, to make note of the fact that:

$$\Gamma^{\mathrm{M}}{}_{\Sigma\mathrm{T},5} = \tfrac{1}{2} g^{\mathrm{MA}}{}_{,5}\left(g_{\mathrm{A}\Sigma,\mathrm{T}} + g_{\mathrm{TA},\Sigma} - g_{\Sigma\mathrm{T},\mathrm{A}}\right) + \tfrac{1}{2} g^{\mathrm{MA}}\left(g_{\mathrm{A}\Sigma,\mathrm{T},5} + g_{\mathrm{TA},\Sigma,5} - g_{\Sigma\mathrm{T},\mathrm{A},5}\right) = 0\,. \tag{5.16}$$

This makes use of (5.6) and the fact that ordinary derivatives commute. A further variation of (5.16) employs (5.1) to also write, for the field strength tensor:

$$\Gamma^{\mathrm{M}}{}_{5\Sigma,5} = \overline{\kappa} F^{\mathrm{M}}{}_{\Sigma,5} = 0\,. \tag{5.17}$$

Again, at bottom, every result in this section is a consequence of relationship (5.1), taken in combination with the antisymmetric field strength $F^{\mathrm{MN}} \equiv -F^{\mathrm{NM}}$. Now, we turn to the Riemann tensor, and Maxwell’s equations.

6. Maxwell’s Equations as Pure Geometry

We have shown how Lorentz force motion might be described as simple geodesic motion in a five-dimensional Kaluza-Klein spacetime. But equations of motion are only one part of a complete field theory. The other part is a specification of how the “sources” of that theory influence the “fields” originating from those sources. In a complete theory, the equations of motion then describe motion through the fields originating from the sources. It is now time to place Maxwell's equations on a firm geometric footing.

In five dimensions, we of course specify the Riemann tensor in the usual way, albeit with an extra axial index. That is:

$$R^{\mathrm{A}}{}_{\mathrm{BMN}} = -\Gamma^{\mathrm{A}}{}_{\mathrm{BM,N}} + \Gamma^{\mathrm{A}}{}_{\mathrm{BN,M}} + \Gamma^{\Sigma}{}_{\mathrm{BN}}\Gamma^{\mathrm{A}}{}_{\Sigma\mathrm{M}} - \Gamma^{\Sigma}{}_{\mathrm{BM}}\Gamma^{\mathrm{A}}{}_{\Sigma\mathrm{N}} . \tag{6.1}$$

Now, let's consider the $\mathrm{M}=5$ component of this equation, that is:

$$R^{\mathrm{A}}{}_{\mathrm{B5N}} = -\Gamma^{\mathrm{A}}{}_{\mathrm{B5,N}} + \Gamma^{\mathrm{A}}{}_{\mathrm{BN,5}} + \Gamma^{\Sigma}{}_{\mathrm{BN}}\Gamma^{\mathrm{A}}{}_{\Sigma 5} - \Gamma^{\Sigma}{}_{\mathrm{B5}}\Gamma^{\mathrm{A}}{}_{\Sigma\mathrm{N}} . \tag{6.2}$$

By virtue of $\Gamma^{\mathrm{M}}{}_{\Sigma\mathrm{T},5}=0$, equation (5.16), which is in turn a consequence of $g_{\mathrm{MN},5}=0$, which is in turn a consequence of $F^{\mathrm{MN}} \equiv -F^{\mathrm{NM}}$, the second term zeros out, and (6.2) becomes:

$$R^{\mathrm{A}}{}_{\mathrm{B5N}} = -\Gamma^{\mathrm{A}}{}_{\mathrm{B5,N}} + \Gamma^{\Sigma}{}_{\mathrm{BN}}\Gamma^{\mathrm{A}}{}_{\Sigma 5} - \Gamma^{\Sigma}{}_{\mathrm{B5}}\Gamma^{\mathrm{A}}{}_{\Sigma\mathrm{N}} . \tag{6.3}$$

Substituting (5.1), i.e., $\Gamma^{\mathrm{M}}{}_{5\Sigma} = \overline{\kappa} F^{\mathrm{M}}{}_{\Sigma}$ into the above, and with some minor term rearrangement, we immediately arrive at the *very critical expression*:

$$R^{\mathrm{A}}{}_{\mathrm{B5N}} = -\overline{\kappa}\left(F^{\mathrm{A}}{}_{\mathrm{B,N}} + \Gamma^{\mathrm{A}}{}_{\Sigma\mathrm{N}}F^{\Sigma}{}_{\mathrm{B}} - \Gamma^{\Sigma}{}_{\mathrm{BN}}F^{\mathrm{A}}{}_{\Sigma}\right) = -\overline{\kappa}F^{\mathrm{A}}{}_{\mathrm{B;N}} . \tag{6.4}$$

In particular, these three remaining terms of $R^{\mathrm{A}}{}_{\mathrm{B5N}}$ *turn out to be identical with the expression for the gravitationally-covariant derivative* $F^{\mathrm{A}}{}_{\mathrm{B;N}}$ *of the mixed field strength tensor*, times the constant factor $-\overline{\kappa}$. This will lead us immediately to a geometric foundation for Maxwell's equations in the following way:

As regards *Maxwell's electric charge equation*, we contract (6.4) down to its Ricci tensor component and define a five-current J_{B} with covariant 5-space index:

$$R_{\mathrm{B5}} = R^{\Sigma}{}_{\mathrm{B5}\Sigma} = -\overline{\kappa}\left(F^{\Sigma}{}_{\mathrm{B},\Sigma} + \Gamma^{\Sigma}{}_{\mathrm{T}\Sigma}F^{\mathrm{T}}{}_{\mathrm{B}} - \Gamma^{\mathrm{T}}{}_{\mathrm{B}\Sigma}F^{\Sigma}{}_{\mathrm{T}}\right) = -\overline{\kappa}F^{\Sigma}{}_{\mathrm{B};\Sigma} \equiv -\overline{\kappa}J_{\mathrm{B}} . \tag{6.5}$$

We will now want to see how J_{B} relates to the observed four-current $j_{\beta} = F^{\sigma}{}_{\beta;\sigma}$ of electrodynamics. We first expand the Σ and T indexes into spacetime and axial parts, and use $\Gamma^{5}{}_{\mathrm{T5}}=0$ and $F^{5}{}_{\mathrm{T}}=0$ from (5.14) to zero out some terms (but *not* any of the covariant derivatives, for reasons to soon become apparent), to obtain:

$$R_{\mathrm{B5}} = -\overline{\kappa}\left(F^{\sigma}{}_{\mathrm{B},\sigma} + \Gamma^{\sigma}{}_{\tau\sigma}F^{\tau}{}_{\mathrm{B}} - \Gamma^{\tau}{}_{\mathrm{B}\sigma}F^{\sigma}{}_{\tau}\right) = -\overline{\kappa}\left(F^{\sigma}{}_{\mathrm{B};\sigma} + F^{5}{}_{\mathrm{B;5}}\right) \equiv -\overline{\kappa}J_{\mathrm{B}} . \tag{6.6}$$

Now, we split the above into two equations, namely:

$$R_{\beta 5} = -\overline{\kappa}\left(F^{\sigma}{}_{\beta,\sigma} + \Gamma^{\sigma}{}_{\tau\sigma}F^{\tau}{}_{\beta} - \Gamma^{\tau}{}_{\beta\sigma}F^{\sigma}{}_{\tau}\right) = -\overline{\kappa}\left(F^{\sigma}{}_{\beta;\sigma} + F^{5}{}_{\beta;5}\right) \equiv -\overline{\kappa}J_{\beta}\text{, and} \tag{6.7}$$

$$R_{55} = -\overline{\kappa}\left(F^{\sigma}{}_{5,\sigma} + \Gamma^{\sigma}{}_{\tau\sigma}F^{\tau}{}_{5} - \Gamma^{\tau}{}_{5\sigma}F^{\sigma}{}_{\tau}\right) = -\overline{\kappa}\left(F^{\sigma}{}_{5;\sigma} + F^{5}{}_{5;5}\right) \equiv -\overline{\kappa}J_{5}\,. \tag{6.8}$$

In (6.7), we discern the four-covariant derivative $F^{\sigma}{}_{\beta;\sigma} = F^{\sigma}{}_{\beta,\sigma} + \Gamma^{\sigma}{}_{\tau\sigma}F^{\tau}{}_{\beta} - \Gamma^{\tau}{}_{\beta\sigma}F^{\sigma}{}_{\tau}$, which means that $F^{5}{}_{\beta;5} = 0$ and that $J_{\beta} = j_{\beta}$ is the observed electromagnetic current source density. We may therefore reduce (6.7) to:

$$R_{\beta 5} = -\overline{\kappa}F^{\sigma}{}_{\beta;\sigma} \equiv -\overline{\kappa}j_{\beta}\,. \tag{6.9}$$

This is Maxwell's electric charge equation, on a geometric foundation.

For the axial equation (6.8), we use $F^{\Sigma}{}_{5} = 0$ to reduce terms as before, but we also employ the substitution $\Gamma^{\mathrm{M}}{}_{5\Sigma} = \overline{\kappa}F^{\mathrm{M}}{}_{\Sigma}$ from (5.1). Thus:

$$R_{55} = -\overline{\kappa}^{2}F^{\tau}{}_{\sigma}F^{\sigma}{}_{\tau} = -\overline{\kappa}\left(F^{\sigma}{}_{5;\sigma} + F^{5}{}_{5;5}\right) \equiv -\overline{\kappa}J_{5} \equiv -\overline{\kappa}j_{5} \neq 0\,. \tag{6.10}$$

Interestingly, despite the $F^{\sigma}{}_{5;\sigma} + F^{5}{}_{5;5}$ term containing two mixed tensors which both vanish in their own right, this term for R_{55} is *not* equal to zero. Rather, we find that the covariant derivative term $F^{\sigma}{}_{5;\sigma} + F^{5}{}_{5;5}$ *does not* vanish, and in fact, leaves a very central term $F^{\sigma\tau}F_{\sigma\tau}$ found in the QED free-field Lagrangian $\mathfrak{L}_{QCD\,(Free)} = -\frac{1}{4}F^{\sigma\tau}F_{\sigma\tau}$ and in the Maxwell stress-energy tensor $T^{\mu}{}_{\nu\,Maxwell} = -\left(F^{\mu\sigma}F_{\nu\sigma} - \frac{1}{4}\delta^{\mu}{}_{\nu}F^{\sigma\tau}F_{\sigma\tau}\right)$ in Heaviside-Lorentz units. Contrasting (6.10) with (6.9), it is apparent that $F^{5}{}_{5;5} = 0$, but that $F^{\sigma}{}_{5;\sigma} = F^{\sigma\tau}F_{\sigma\tau} \neq 0$. This is the first of several instances where we will find that a covariant derivative of $F^{\mathrm{T}}{}_{5} = 0$ or its covariant and contravariant relatives, is non-zero. One may think of $F^{\sigma}{}_{5;\sigma} = F^{\sigma\tau}F_{\sigma\tau} \neq 0$ as being "gravitationally induced" out of $F^{\sigma}{}_{5} = 0$, solely as a non-linear gravitational effect, because in the absence of gravitation, covariant derivatives approach ordinary derivatives and so $F^{\sigma}{}_{5;\sigma} \rightarrow F^{\sigma}{}_{5,\sigma} = 0$. Consolidating (6.9) and (6.10) together for contrast, we see that the five-vector for $R_{\mathrm{B}5}$ is given by:

$$\begin{cases} R_{\beta 5} = -\overline{\kappa} F^{\sigma}{}_{\beta;\sigma} = -\overline{\kappa} j_{\beta} = -\overline{\kappa}\left(F^{\sigma}{}_{\beta,\sigma} + \Gamma^{\sigma}{}_{\tau\sigma} F^{\tau}{}_{\beta} - \Gamma^{\tau}{}_{\beta\sigma} F^{\sigma}{}_{\tau}\right) \\ R_{55} = -\overline{\kappa} F^{\sigma}{}_{5;\sigma} = -\overline{\kappa} j_5 \qquad\qquad\qquad = -\overline{\kappa}^2 F^{\sigma\tau} F_{\sigma\tau} \end{cases}, \tag{6.11}$$

which we consolidate to:

$$R_{\mathrm{B}5} = -\overline{\kappa} j_{\mathrm{B}} = -\overline{\kappa} F^{\sigma}{}_{\mathrm{B};\sigma}. \tag{6.12}$$

The other consolidated relationship, emerging from $F^{5}{}_{\beta;5} = 0$ and $F^{5}{}_{5;5} = 0$, is:

$$F^{5}{}_{\mathrm{B};5} = 0. \tag{6.13}$$

We make a special point of this, because when we consider the Ricci tensor in mixed form, e.g., $R^{\mathrm{B}}{}_{5}$, we will find that the terms analogous to (6.13) become non-zero as well (just like $F^{\sigma}{}_{5;\sigma}$ in (6.11)), and contribute an "axial" component to the currents which may help to resolve the chirality problems often found in Kaluza-Klein theory. It is already worth noting from (6.11), that $j_{\beta} = \overline{\psi}\gamma_{\beta}\psi$ is a vector current, so that $j_5 \propto \overline{\psi}\gamma_5\psi$ will most certainly be a pseudoscalar. We use the proportionality for j_5, because we have not chosen a representation of the 5-D Clifford algebra $\frac{1}{2}\{\Gamma_{\mathrm{M}}, \Gamma_{\mathrm{N}}\} \equiv g_{\mathrm{MN}}$, and although the intrinsic spin results of section 4 herein seem to lean spacelike, this exact choice of representation does depend upon whether the fifth dimension is timelike or spacelike, see., e.g., [10], section 3.

Turning now to Maxwell's magnetic equation, we first lower the A index in (6.4), and use $R_{\mathrm{ABMN}} = R_{\mathrm{MNAB}}$ to write:

$$R_{5\mathrm{NMB}} = g_{\mathrm{MA}} R^{\mathrm{A}}{}_{\mathrm{B5N}} = -\overline{\kappa}\left(g_{\mathrm{MA}} F^{\mathrm{A}}{}_{\mathrm{B,N}} + g_{\mathrm{MA}} \Gamma^{\mathrm{A}}{}_{\Sigma\mathrm{N}} F^{\Sigma}{}_{\mathrm{B}} - g_{\mathrm{MA}} \Gamma^{\Sigma}{}_{\mathrm{BN}} F^{\mathrm{A}}{}_{\Sigma}\right) = -\overline{\kappa} g_{\mathrm{MA}} F^{\mathrm{A}}{}_{\mathrm{B;N}}. \tag{6.14}$$

Maxwell's magnetic equation then arises straight from the 5-dimensional rendition of the "first" Bianchi identity:

$$R_{\mathrm{MNAB}} + R_{\mathrm{MABN}} + R_{\mathrm{MBNA}} = 0. \tag{6.15}$$

Making use of (6.14), the $\mathrm{M} = 5$ component of this is:

$$R_{5\mathrm{NAB}} + R_{5\mathrm{ABN}} + R_{5\mathrm{BNA}} = -\overline{\kappa}\left(F_{\mathrm{AB;N}} + F_{\mathrm{BN;A}} + F_{\mathrm{NA;B}}\right) = -\overline{\kappa}\left(F_{\mathrm{AB,N}} + F_{\mathrm{BN,A}} + F_{\mathrm{NA,B}}\right) = 0, \tag{6.16}$$

where we account for the well-known fact that in the cyclic combination of (6.16) with antisymmetric tensors, the Christoffel terms in the covariant derivatives cancel identically, so the covariant derivatives becomes ordinary derivatives. In the $\mathrm{NAB} = \eta\alpha\beta$ subset of this, we immediately obtain Maxwell's magnetic equation

$$F_{\alpha\beta,\nu} + F_{\beta\nu,\alpha} + F_{\nu\alpha,\beta} = 0\,. \tag{6.17}$$

In light of our earlier discovery of some new terms in Maxwell's electric charge equation arising from the fifth dimension, see, e.g., the R_{55} equation in (6.11), one may ask whether there are any additional electrodynamic terms in the (6.16) above, in the circumstance where more than a single axial index is employed. Because $R_{\mathrm{ABMN}} = R_{\mathrm{MNAB}} = -R_{\mathrm{BAMN}}$, it is clear that with more than two axial indexes, i.e., $R_{555\mu}$, (6.16) will identically reduce to zero. But we should explore whether there is any additional electrodynamic information to be gleaned when exactly two axial indexes are used in (6.16). Thus, we may examine, say:

$$R_{55\mathrm{AB}} + R_{5\mathrm{AB}5} + R_{5\mathrm{B}5\mathrm{A}} = -\overline{\kappa}\left(F_{\mathrm{AB};5} + F_{\mathrm{B}5;\mathrm{A}} + F_{5\mathrm{A};\mathrm{B}}\right) = -\overline{\kappa}\left(F_{\mathrm{AB},5} + F_{\mathrm{B}5,\mathrm{A}} + F_{5\mathrm{A},\mathrm{B}}\right) = 0\,. \tag{6.18}$$

We learn from (6.11), especially $R_{55} = -\overline{\kappa} F^{\sigma}{}_{5;\sigma} = -\overline{\kappa}^{2} F^{\sigma\tau} F_{\sigma\tau}$, not to automatically eliminate a field strength term such as $F^{\sigma}{}_{5}$ when it appears in a *covariant* derivative, i.e., $F^{\sigma}{}_{5;\sigma}$. However, the migration of covariant to ordinary derivatives in the cyclic combination of (6.16) removes this complication. We know from (5.11) that $F_{\mathrm{B}5} = F_{5\mathrm{A}} = 0$ as so their *ordinary* derivatives will vanish as well. The remaining term $F_{\mathrm{AB},5} = \left(g_{\mathrm{A}\Sigma} F^{\Sigma}{}_{\mathrm{B}}\right)_{,5} = g_{\mathrm{A}\Sigma,5} F^{\Sigma}{}_{\mathrm{B}} + g_{\mathrm{A}\Sigma} F^{\Sigma}{}_{\mathrm{B},5} = 0$ in (6.18), by virtue of (5.6) and (5.17). Thus, (6.18) is identically equal to zero, not only because of the Bianchi identity, but because of the inherent properties of the F_{AB} and g_{AB} developed in section 5. Thus, there is no additional electrodynamic information to be gleaned from (6.18).

We have now placed each of Maxwell's equations on a solely geometric footing. Maxwell's source equation in covariant (lower index) form is specified by (6.9), namely, $R_{\beta 5} = -\overline{\kappa} j_{\beta} = -\overline{\kappa} F^{\sigma}{}_{\beta;\sigma}$, and there is an additional component in the 5-dimensional space given by the latter of (6.11), namely, $R_{55} = -\overline{\kappa} j_{5} = -\overline{\kappa} F^{\sigma}{}_{5;\sigma} = -\overline{\kappa}^{2} F^{\sigma\tau} F_{\sigma\tau}$, which contains the very central term $\mathfrak{L}_{QCD\,(Free)} = -\frac{1}{4} F^{\sigma\tau} F_{\sigma\tau}$, and which will be of great interest in the discussion to follow. Maxwell's magnetic equation is simply an axial component (6.16) of the first Bianchi identity. And, the Lorentz force equation (2.6), upon which the foregoing geometrization of

Maxwell's equations is based, is no more and no less than equation (2.4) for geodesic motion in the five-dimensional geometry. With source equations producing fields and with material bodies in those fields moving over geodesics that are identical to and synonymous with the Lorentz force, Maxwell's electrodynamics now rests on the firm geometrodynamic footing of a five-dimensional Kaluza-Klein geometry.

7. Calculation of the 5-Dimensional Curvature Scalar, and the Fifth-Dimensional Components of the Einstein Equation.

Especially in light of the gravitationally-induced $R_{55} = -\frac{1}{16}\overline{\kappa}^2 F^{\sigma\tau}F_{\sigma\tau}$, see (6.11), we now turn our attention to the QED Lagrangian density $\hbar c^2 \mathfrak{L}_{QED} = \left(-F^{\sigma\tau}F_{\sigma\tau} - A_\mu j^\mu\right)$, and in particular, to seeing if we can place this entire $\mathfrak{L}_{QED}$ *with sources*, on a purely geometric footing, *in vacuo*. In other words, we are now starting to take aim, as discussed in the introduction, at using an Einstein-Hilbert Action of the general form $S = \int kR dV$ to specify $\mathfrak{L}_{QED}$ with sources, but without explicitly adding an $\mathfrak{L}_{\mathrm{Matter}}$. We begin discussion here by deriving the five-dimensional Ricci curvature scalar $R_{(5)} \equiv R^\Sigma{}_\Sigma = R + R^5{}_5$, taking the four dimensional curvature scalar to be $R = R^\sigma{}_\sigma$, since these are the clear candidates for inclusion in such an action. In addition, we need $R_{(5)}$ if we wish to consider the five-dimensional extensions of Einstein's equation, i.e., $-\kappa T^{\mathrm{M}}{}_{\mathrm{N}} = R^{\mathrm{M}}{}_{\mathrm{N}} - \frac{1}{2}\delta^{\mathrm{M}}{}_{\mathrm{N}} R_{(5)}$.

There are two ways to calculate $R^5{}_5$ which lead to alternative, but equivalent expressions. First, in (6.5), we have already found $R_{\mathrm{B}5}$. So, all we need do is raise the index using $R^{\mathrm{M}}{}_5 = g^{\mathrm{MB}} R_{\mathrm{B}5}$, i.e.,

$$R^{\mathrm{M}}{}_5 = -\overline{\kappa}\left(g^{\mathrm{MB}} F^\Sigma{}_{\mathrm{B},\Sigma} + \Gamma^\Sigma{}_{\mathrm{T}\Sigma} F^{\mathrm{TM}} - g^{\mathrm{MB}} \Gamma^{\mathrm{T}}{}_{\mathrm{B}\Sigma} F^\Sigma{}_{\mathrm{T}}\right) = -\overline{\kappa} g F^{\Sigma\mathrm{M}}{}_{;\Sigma} \equiv -\overline{\kappa} J^{\mathrm{M}}, \tag{7.1}$$

and then take the $\mathrm{M} = 5$ component. Second, alternatively, we can write the covariant Ricci tensor as $R_{\mathrm{MN}} = g_{\mathrm{M}\Sigma} R^\Sigma{}_{\mathrm{N}} = g_{\mathrm{M}\sigma} R^\sigma{}_{\mathrm{N}} + g_{\mathrm{M}5} R^5{}_{\mathrm{N}}$, then take the $R_{55} = g_{5\sigma} R^\sigma{}_5 + g_{55} R^5{}_5$ component, which we rewrite as:

$$g_{55} R^5{}_5 = R_{55} - g_{5\sigma} R^\sigma{}_5 . \tag{7.2}$$

In this latter approach, we can take advantage of the fact that $g_{55} = \pm 1 = \text{constant}$ depending on whether the fifth dimension is timelike (+1) or spacelike (-1), see the discussion following (5.7), and can make use of $R_{55} = -\overline{\kappa}^2 F^{\sigma\tau} F_{\sigma\tau}$, as already found in (6.10). In either approach, since the unknowns in (7.2) are $R^{\sigma}{}_5$ and $g_{5\sigma}$, the first step is to deduce $R^{\mathrm{M}}{}_5$ in (7.1).

Starting from (7.1), we separate all contracted indexes into their spacetime and axial components. We can reduce many terms throughout making use of $F^{\Sigma}{}_5 = 0$ and its raised and lowered variants, as well as $\Gamma^5{}_{\tau 5} = 0$, see (5.14). Along the way, we also use (5.1) to substitute $\Gamma^{\mathrm{T}}{}_{5\Sigma} = \overline{\kappa} F^{\mathrm{T}}{}_{\Sigma}$. This introduces another "gravitationally-induced" term $-\overline{\kappa} g^{\mathrm{M}5} F^{\mathrm{T}}{}_{\Sigma} F^{\Sigma}{}_{\mathrm{T}}$ as in the second equation (6.11), which had no counterpart in the covariant-indexed $R_{\mathrm{B}5}$ of (6.5). Because $F^{\mathrm{T}}{}_{\Sigma} F^{\Sigma}{}_{\mathrm{T}}$ is summed over all five dimensions, we can readjust the indexes according to $F^{\mathrm{T}}{}_{\Sigma} F^{\Sigma}{}_{\mathrm{T}} = F^{\mathrm{T}\Sigma} F_{\Sigma\mathrm{T}} = -F^{\Sigma\mathrm{T}} F_{\Sigma\mathrm{T}}$. Finally, recalling the "gravitationally-induced" term $-\overline{\kappa} F^{\sigma}{}_{5;\sigma} = -\overline{\kappa}^2 F^{\sigma\tau} F_{\sigma\tau} \neq 0$ from (6.11), we use $F^{\Sigma}{}_5 = 0$ to eliminate *only ordinary derivatives* such as $F^{\Sigma}{}_{5,\mathrm{T}} = 0$, but *not the covariant derivatives* $F^{\Sigma}{}_{5;\mathrm{T}}$. The net result of all of this, is that (7.1) reduces to:

$$\begin{aligned} R^{\mathrm{M}}{}_5 &= -\overline{\kappa}\left(g^{\mathrm{M}\beta} F^{\sigma}{}_{\beta,\sigma} + \Gamma^{\sigma}{}_{\tau\sigma} F^{\tau\mathrm{M}} - g^{\mathrm{M}\beta} \Gamma^{\tau}{}_{\beta\sigma} F^{\sigma}{}_{\tau} + \overline{\kappa} g^{\mathrm{M}5} F^{\sigma\tau} F_{\sigma\tau} \right) \\ &= -\overline{\kappa} F^{\Sigma\mathrm{M}}{}_{;\Sigma} = -\overline{\kappa}\left(F^{\sigma\mathrm{M}}{}_{;\sigma} + F^{5\mathrm{M}}{}_{;5} \right) \equiv -\overline{\kappa} J^{\mathrm{M}} \equiv -\overline{\kappa}\left(j^{\mathrm{M}} + j_{(5)}{}^{\mathrm{M}} \right) \end{aligned} . \tag{7.3}$$

In the above, we define an "ordinary" 5-dimensional current

$$j^{\mathrm{M}} \equiv F^{\sigma\mathrm{M}}{}_{;\sigma} = g^{\mathrm{M}\beta} F^{\sigma}{}_{\beta,\sigma} + \Gamma^{\sigma}{}_{\tau\sigma} F^{\tau\mathrm{M}} - g^{\mathrm{M}\beta} \Gamma^{\tau}{}_{\beta\sigma} F^{\sigma}{}_{\tau} , \tag{7.4}$$

as well as a "gravitationally-induced" five-dimensional current:

$$j_{(5)}{}^{\mathrm{M}} \equiv F^{5\mathrm{M}}{}_{;5} = \overline{\kappa} g^{\mathrm{M}5} F^{\sigma\tau} F_{\sigma\tau} \neq 0 . \tag{7.5}$$

Returning with hindsight to (6.13) for which we define $j_{(5)\mathrm{B}} \equiv F^5{}_{\mathrm{B};5} = 0$, we see now that $j_{(5)}{}^{\mathrm{M}}$ is zero in its covariant (lower index) form, but is "induced" to be non-zero when raised into contravariant form. We noted earlier that $j_{\beta} = \overline{\psi}\gamma_{\beta}\psi$ is a vector current and $j_5 \propto \overline{\psi}\gamma_5\psi$ a pseudoscalar. We raise the question, without exploration at this time, whether $j_{(5)\beta} \propto \overline{\psi}\gamma_{\beta}\gamma_5\psi = 0$ and $j_{(5)5} \propto \overline{\psi}\gamma_5\gamma_5\psi \propto \overline{\psi}\psi = 0$ are also axial vector and pure mass-term $\overline{\psi}\psi$

currents which are zero in covariant form, but are induced to become non-zero when they are raised into contravariant form, and whether this might yield a path to solving the chirality problem of five-dimensional Kaluza-Klein theories.

Returning to our present task, which is to calculate $R_{(5)}$ in two alternative ways, let's now separate (7.3) into two separate equations:

$$\begin{aligned} R^{\mu}{}_{5} &= -\overline{\kappa}\left(g^{\mu\beta}F^{\sigma}{}_{\beta,\sigma} + \Gamma^{\sigma}{}_{\tau\sigma}F^{\tau\mu} - g^{\mu\beta}\Gamma^{\tau}{}_{\beta\sigma}F^{\sigma}{}_{\tau} + \overline{\kappa}g^{\mu5}F^{\sigma\tau}F_{\sigma\tau}\right) \\ &= -\overline{\kappa}F^{\Sigma\mu}{}_{;\Sigma} = -\overline{\kappa}\left(F^{\sigma\mu}{}_{;\sigma} + F^{5\mu}{}_{;5}\right) \equiv -\overline{\kappa}J^{\mu} \equiv -\overline{\kappa}\left(j^{\mu} + j_{(5)}{}^{\mu}\right) \end{aligned}, \text{ and} \tag{7.6}$$

$$\begin{aligned} R^{5}{}_{5} &= -\overline{\kappa}\left(g^{5\beta}F^{\sigma}{}_{\beta,\sigma} - g^{5\beta}\Gamma^{\tau}{}_{\beta\sigma}F^{\sigma}{}_{\tau} + \overline{\kappa}g^{55}F^{\sigma\tau}F_{\sigma\tau}\right) \\ &= -\overline{\kappa}F^{\Sigma5}{}_{;\Sigma} = -\overline{\kappa}\left(F^{\sigma5}{}_{;\sigma} + F^{55}{}_{;5}\right) \equiv -\overline{\kappa}J^{5} \equiv -\overline{\kappa}\left(j^{5} + j_{(5)}{}^{5}\right) \end{aligned}, \tag{7.7}$$

where we use $F^{\tau5} = 0$ to eliminate one term from (7.7). The foregoing contain four distinct current types, referenced above in relation to the chirality discussion, explicitly written as:

$$j^{\mu} = F^{\sigma\mu}{}_{;\sigma} = g^{\mu\beta}F^{\sigma}{}_{\beta,\sigma} + \Gamma^{\sigma}{}_{\tau\sigma}F^{\tau\mu} - g^{\mu\beta}\Gamma^{\tau}{}_{\beta\sigma}F^{\sigma}{}_{\tau}. \tag{7.8}$$

$$j_{(5)}{}^{\mu} = F^{5\mu}{}_{;5} = \overline{\kappa}g^{\mu5}F^{\sigma\tau}F_{\sigma\tau}. \tag{7.9}$$

$$j^{5} = F^{\sigma5}{}_{;\sigma} = g^{5\beta}F^{\sigma}{}_{\beta,\sigma} - g^{5\beta}\Gamma^{\tau}{}_{\beta\sigma}F^{\sigma}{}_{\tau}. \tag{7.10}$$

$$j_{(5)}{}^{5} = F^{55}{}_{;5} = \overline{\kappa}g^{55}F^{\sigma\tau}F_{\sigma\tau}. \tag{7.11}$$

So, now we can write out the five-dimensional curvature scalar $R_{(5)} = R + R^{5}{}_{5}$, leaving R as a remaining unknown still to be deduced. The first way to do this, directly from a rearranged (7.7), is to write:

$$R_{(5)} = R + R^{5}{}_{5} = R - \overline{\kappa}^{2}g^{55}F^{\sigma\tau}F_{\sigma\tau} - \overline{\kappa}j^{5}. \tag{7.12}$$

The second way to do this, based on (7.2) and using $R_{55} = -\overline{\kappa}^{2}F^{\sigma\tau}F_{\sigma\tau}$ from (6.10), and (7.6) rearranged into $R^{\mu}{}_{5} = -\overline{\kappa}\left(j^{\mu} + \overline{\kappa}g^{\mu5}F^{\sigma\tau}F_{\sigma\tau}\right)$, and $R_{(5)} = R + R^{5}{}_{5}$ multiplied through by g_{55} into $g_{55}R_{(5)} = g_{55}R + g_{55}R^{5}{}_{5}$, is:

$$g_{55}R_{(5)} = g_{55}R + g_{55}R^{5}{}_{5} = g_{55}R + R_{55} - g_{5\mu}R^{\mu}{}_{5} = g_{55}R - \overline{\kappa}^{2}F^{\sigma\tau}F_{\sigma\tau} + \overline{\kappa}g_{5\mu}\left(j^{\mu} + \overline{\kappa}g^{\mu5}F^{\sigma\tau}F_{\sigma\tau}\right). \tag{7.13}$$

This last expression can, however, be reduced using $g_{5\mu}g^{\mu 5}=0$, see (5.8), down to:

$$g_{55}R_{(5)} = g_{55}R - \overline{\kappa}^2 F^{\sigma\tau}F_{\sigma\tau} + \overline{\kappa}g_{5\mu}j^{\mu} . \tag{7.14}$$

Keep in mind that $g_{55}=\pm 1$, depending on whether the fifth dimension is timelike or spacelike.

Equations (7.12) and (7.14) are totally-equivalent expressions, and they are each of interest in different circumstances. Equation (7.14) is of interest, because it appears to resemble the QED Lagrangian $\hbar c^2 \mathcal{L}_{QED} = \left(-\frac{1}{4}F^{\sigma\tau}F_{\sigma\tau} - A_\mu j^\mu\right)$, and may provide a direct basis for geometrically representing $\mathcal{L}_{QED}$, if we can make a suitable association between $g_{5\mu}$ and A_μ, each of which is a dynamical field, to use in the term $g_{5\mu}j^\mu$. So, in the next section, we will explicitly explore the connection between the gravitational potentials $g_{5\mu}$ and the electrodynamic potentials A_μ. However, first, it behooves us to calculates the axial components of Einstein's equation generalized to five dimensions: $-\kappa T^{\mathrm{M}}{}_{\mathrm{N}} = R^{\mathrm{M}}{}_{\mathrm{N}} - \frac{1}{2}\delta^{\mathrm{M}}{}_{\mathrm{N}}R_{(5)}$, and here, (7.12) is the preferred expression.

To calculate the axial components of $-\kappa T^{\mathrm{M}}{}_{\mathrm{N}} = R^{\mathrm{M}}{}_{\mathrm{N}} - \frac{1}{2}\delta^{\mathrm{M}}{}_{\mathrm{N}}R_{(5)}$, we use (7.3) and (7.12) to write:

$$-\kappa T^{\mathrm{M}}{}_5 = R^{\mathrm{M}}{}_5 - \tfrac{1}{2}\delta^{\mathrm{M}}{}_5 R_{(5)} = -\overline{\kappa}\left(j^{\mathrm{M}} + j_{(5)}{}^{\mathrm{M}}\right) - \tfrac{1}{2}\delta^{\mathrm{M}}{}_5\left(R - \overline{\kappa}^2 g^{55}F^{\sigma\tau}F_{\sigma\tau} - \overline{\kappa}j^5\right). \tag{7.15}$$

This splits into two equations: $j_{(5)}{}^{\mu} = F^{5\mu}{}_{;5} = \overline{\kappa}g^{\mu 5}F^{\sigma\tau}F_{\sigma\tau}$ (7.9)

$$-\kappa T^{\mu}{}_5 = R^{\mu}{}_5 = -\overline{\kappa}\left(j^\mu + j_{(5)}{}^{\mu}\right) = -\overline{\kappa}\left(j^\mu + \overline{\kappa}g^{\mu 5}F^{\sigma\tau}F_{\sigma\tau}\right);\text{ and} \tag{7.16}$$

$$-\kappa T^5{}_5 = R^5{}_5 - \tfrac{1}{2}R_{(5)} = -\tfrac{1}{2}\overline{\kappa}\left(j^5 + j_{(5)}{}^5\right) - \tfrac{1}{2}R = -\tfrac{1}{2}\overline{\kappa}\left(j^5 + \overline{\kappa}g^{55}F^{\sigma\tau}F_{\sigma\tau}\right) - \tfrac{1}{2}R , \tag{7.17}$$

where we have employed (7.9) to consolidate the former and (7.11) for the latter. The four-dimensional Ricci scalar R is still an unknown in (7.17) and elsewhere; in the next section, we will see how to further deduce this this.

8. The Vector Potential, the Gravitational Potential, and the Exact QED Lagrangian

Once again, we start with (5.1), written out as (recall $g_{\Sigma\mathrm{T},5}=0$, see (5.6)):

$$\bar{\kappa}F^{\mathrm{M}}{}_{\mathrm{T}} = \Gamma^{\mathrm{M}}{}_{\mathrm{T5}} = \tfrac{1}{2} g^{\mathrm{MA}}\left(g_{\mathrm{AT,5}} + g_{\mathrm{5A,T}} - g_{\mathrm{T5,A}}\right) = \tfrac{1}{2} g^{\mathrm{MA}}\left(g_{\mathrm{5A,T}} - g_{\mathrm{5T,A}}\right). \tag{8.1}$$

It is helpful to lower the indexes in field strength tensor and connect this to the covariant vector potentials A_μ, generalized into 5-dimensions as A_{M} via $F_{\Sigma\mathrm{T}} \equiv A_{\Sigma;\mathrm{T}} - A_{\mathrm{T};\Sigma} = A_{\Sigma,\mathrm{T}} - A_{\mathrm{T},\Sigma}$, as such:

$$\bar{\kappa}\left(A_{\Sigma;\mathrm{T}} - A_{\mathrm{T};\Sigma}\right) = \bar{\kappa}F_{\Sigma\mathrm{T}} = \bar{\kappa}g_{\Sigma\mathrm{M}}F^{\mathrm{M}}{}_{\mathrm{T}} = \tfrac{1}{2} g_{\Sigma\mathrm{M}}g^{\mathrm{MA}}\left(g_{\mathrm{5A,T}} - g_{\mathrm{5T,A}}\right) = \tfrac{1}{2}\left(g_{5\Sigma,\mathrm{T}} - g_{5\mathrm{T},\Sigma}\right). \tag{8.2}$$

The relationship $\bar{\kappa}F_{\Sigma\mathrm{T}} = \bar{\kappa}\left(A_{\Sigma;\mathrm{T}} - A_{\mathrm{T};\Sigma}\right) = \tfrac{1}{2}\left(g_{5\Sigma,\mathrm{T}} - g_{5\mathrm{T},\Sigma}\right)$ expresses clearly, the antisymmetry of $F_{\Sigma\mathrm{T}}$ in terms of the remaining connection terms involving the gravitational potential. Of particular interest, is that we may extract from (8.2), the relation:

$$\bar{\kappa}A_{\Sigma;\mathrm{T}} = \tfrac{1}{2} g_{5\Sigma,\mathrm{T}} = \tfrac{1}{2}\bar{\kappa}h_{5\Sigma,\mathrm{T}}, \tag{8.3}$$

using also $g_{\mathrm{MN}} = \eta_{\mathrm{MN}} + \bar{\kappa}h_{\mathrm{MN}}$ for the gravitational potential energy h_{MN}. If one forms $A_{\Sigma;\mathrm{T}} - A_{\mathrm{T};\Sigma}$ from (8.3) and then renames indexes and uses $g_{\mathrm{MN}} = g_{\mathrm{NM}}$, one arrives back at (8.2). The reason we did not remove the covariant derivative via $F_{\Sigma\mathrm{T}} \equiv A_{\Sigma;\mathrm{T}} - A_{\mathrm{T};\Sigma} = A_{\Sigma,\mathrm{T}} - A_{\mathrm{T},\Sigma}$, is that in (8.3), $A_{\Sigma;\mathrm{T}}$ is considered distinctly from $-A_{\mathrm{T};\Sigma}$, and so the covariant derivatives do not become ordinary unless and until one forms $F_{\Sigma\mathrm{T}} \equiv A_{\Sigma;\mathrm{T}} - A_{\mathrm{T};\Sigma} = A_{\Sigma,\mathrm{T}} - A_{\mathrm{T},\Sigma}$.

Equation (8.3) is a first order differential equation which tells us that the *covariant* derivative of the electrodynamic potential A_Σ is identical with the *ordinary* derivative of the gravitational potential $h_{5\Sigma}$. In the weak field limit, where covariant derivatives become *approximately* equal to ordinary derivatives, $\bar{\kappa}A_{\Sigma;\mathrm{T}} = \tfrac{1}{2} g_{5\Sigma,\mathrm{T}} = \tfrac{1}{2}\bar{\kappa}h_{5\Sigma,\mathrm{T}} \approx \bar{\kappa}A_{\Sigma,\mathrm{T}}$, and so, integrating based on this approximation, we obtain:

$$g_{5\Sigma} = \bar{\kappa}h_{5\Sigma} \approx 2\bar{\kappa}A_\Sigma. \tag{8.4}$$

Now, we return to examine (7.14) *in this weak field limit*, $A_{\Sigma;\mathrm{T}} \approx A_{\Sigma,\mathrm{T}}$. Most importantly, referring to (8.4), the final term in (7.14) becomes $\bar{\kappa}g_{5\mu}j^\mu \approx 2\bar{\kappa}^2 A_\mu j^\mu$. Thus, substituting from (8.4) into (7.14) yields:

and using $\bar{\kappa}^2 = 2\kappa/\hbar c$ yields:

$$g_{55}R_{(5)} \approx g_{55}R + \overline{\kappa}^{2}\left(-F^{\sigma\tau}F_{\sigma\tau} + 2A_{\mu}j^{\mu}\right). \tag{8.5}$$

Now, let's continue with this weak-field limit, to make several further connections of interest, and especially, to deduce the four-dimensional Ricci scalar $R = R^{\sigma}{}_{\sigma}$ which was still unknown in (7.17). Because (8.5) contains $g_{55}R_{(5)}$, the exact expression for $R_{(5)}$ depends upon whether $g_{55} = +1$ (timelike) or $g_{55} = -1$, spacelike.

For a timelike fifth dimension:

$$R_{(5)} \approx R + \overline{\kappa}^{2}\left(-F^{\sigma\tau}F_{\sigma\tau} + 2A_{\mu}j^{\mu}\right). \tag{8.6}$$

For spacelike, (8.6) becomes:

$$-R_{(5)} \approx -R + \overline{\kappa}^{2}\left(-F^{\sigma\tau}F_{\sigma\tau} + 2A_{\mu}j^{\mu}\right). \tag{8.7}$$

Now, up until this point, all of the development has been based on a single supposition introduced just after (2.6): the requirement that the Lorentz force must be represented as nothing other than geodesic motion in a five-dimensional geometry, as expressed in (2.7) and (2.8). Other than perhaps our imposing the requirement that $F^{\mathrm{MN}} \equiv -F^{\mathrm{NM}}$, every step taken since then has been fully deductive, with no other assumptions. We have even left open the question of whether the fifth dimension is timelike or spacelike, simply exploring the consequences in the alternative, as pertinent. This has enabled us to fully specify the axial components of the energy tensor, see (7.15) through (7.17), and to obtain the five dimensional Ricci scalar $R_{(5)}$, up to the four-dimensional scalar $R = R^{\sigma}{}_{\sigma}$ which remains undetermined in (7.17) and (8.5) through (8.7). To deduce $R = R^{\sigma}{}_{\sigma}$, we now must make a new supposition, which we do as follows:

Many authors write the QED Lagrangian density as $\hbar c^{2}\mathfrak{L}_{QED} = \left(-\frac{1}{4}F^{\sigma\tau}F_{\sigma\tau} - A_{\mu}j^{\mu}\right)$ (with $\hbar = c = 1$). However, by rescaling the sign of the source current density, it is equally proper to use the convention $\hbar c^{2}\mathfrak{L}_{QED} = \left(-\frac{1}{4}F^{\sigma\tau}F_{\sigma\tau} + A_{\mu}j^{\mu}\right)$, see, e.g., [12], page 30. By virtue of the opposite signs as between $F^{\sigma\tau}F_{\sigma\tau}$ and $2A_{\mu}j^{\mu}$ in (8.6), and given that there is no choice of the constant factors back in (3.2) and (3.3) which would have reversed this, we shall use this latter convention to write $\mathfrak{L}_{QED}$. Nor would any choice, by the way, have altered the ratio of $-1:2$ between the constant factors multiplying $F^{\sigma\tau}F_{\sigma\tau}$ and $A_\mu j^\mu$, into the $-1:4$ ratio in $\mathfrak{L}_{QED}$.

Now, using this $\hbar c^2\mathfrak{L}_{QED} = \left(-\frac{1}{4}F^{\sigma\tau}F_{\sigma\tau} + A_\mu j^\mu\right)$, the action is formed according to $S(A_\mu) = \int \mathfrak{L}_{QED}\sqrt{-g}\,d^4x$. If, however, we can turn (8.6) and (8.7) into expressions in which the ratio of the constant factors multiplying $F^{\sigma\tau}F_{\sigma\tau}$ and $A_\mu j^\mu$ is $-1:4$ rather than $-1:2$, then we could use these expressions to write QED in terms of a gravitational action, *in vacuo*, of the form $S = \int kRdV$. Because $R = R^\sigma{}_\sigma$ is still an unknown, we shall now use these observations to deduce $R = R^\sigma{}_\sigma$ as such:

We shall select $R = R^\sigma{}_\sigma$ in (8.6) and (8.7) <u>such that</u> the ratio of the constant factors multiplying $F^{\sigma\tau}F_{\sigma\tau}$ and $A_\mu j^\mu$ changes from $-1:2$, to $-1:4$, and also, <u>such that</u> R only contains $F^{\sigma\tau}F_{\sigma\tau}$, and not $A_\mu j^\mu$. Again, these are affirmative requirements, not deductions. We may impose these requirements by rewriting (8.6) and (8.7) as:

$$R_{(5)} \approx R + \overline{\kappa}^2\left(-F^{\sigma\tau}F_{\sigma\tau} + 2A_\mu j^\mu\right) = \overline{\kappa}^2\left(-\tfrac{1}{2}F^{\sigma\tau}F_{\sigma\tau} + 2A_\mu j^\mu\right). \quad (8.8)$$

$$-R_{(5)} \approx -R + \overline{\kappa}^2\left(-F^{\sigma\tau}F_{\sigma\tau} + 2A_\mu j^\mu\right) = +\overline{\kappa}^2\left(-\tfrac{1}{2}F^{\sigma\tau}F_{\sigma\tau} + 2A_\mu j^\mu\right). \quad (8.9)$$

It is then easy to deduce from these, respectively, also using $\frac{\kappa}{\hbar c} = \frac{1}{2}\overline{\kappa}^2 = \frac{8\pi G}{\hbar c^5}$, that:

$$g_{55}R = \tfrac{1}{2}\overline{\kappa}^2 F^{\sigma\tau}F_{\sigma\tau} = \frac{\kappa}{\hbar c}F^{\sigma\tau}F_{\sigma\tau}\text{, and} \quad (8.10)$$

where $g_{55} = \pm 1$ for a timelike (+) and spacelike (-) fifth dimension, respectively. The choice of timelike versus spacelike, merely flips the sign of the (four-dimensional) Ricci scalar.

With $R = R^\sigma{}_\sigma$ now established, we can go back and write the five-dimensional Ricci scalars (8.8) and (8.9), respectively, as: $2\frac{\kappa}{\hbar c} = \overline{\kappa}^2$

$$g_{55}R_{(5)} \approx \overline{\kappa}^2\left(-\tfrac{1}{2}F^{\sigma\tau}F_{\sigma\tau} + 2A_\mu j^\mu\right) = 4\frac{\kappa}{\hbar c}\left(-\tfrac{1}{4}F^{\sigma\tau}F_{\sigma\tau} + A_\mu j^\mu\right) = 4\kappa c\mathfrak{L}_{QED}. \quad (8.11)$$

Apparently, the respective timelike versus spacelike choice also flips the sign for $R_{(5)}$.

Therefore, it become possible to rewrite the QED action $S(A_\mu)=\int \mathfrak{L}_{QED}\sqrt{-g}\,d^4x$ as:

$$S(A_\mu)=\int \mathfrak{L}_{QED}\sqrt{-g}\,d^4x \approx g_{55}\frac{1}{4\kappa c}\int R_{(5)}\sqrt{-g}\,d^4x\,, \tag{8.12}$$

This is the Lagrangian (action) for the vacuum, because it does not contain any explicit matter terms, but only contains $R_{(5)}$. We can put this into words by saying that the QED action is equal to the five-dimensional Ricci scalar, integrated over the four-volume of spacetime. A Ricci scalar derived from all five dimensions, integrated over ordinary spacetime, results in Quantum Electrodynamics. QED is the four-dimensional manifestation of a five-dimensional universe! This achieves the goal set out in the introduction, of generating QED out of an *in vacuo* action of the general form $S=\frac{1}{k}\int R\,dV$, and (8.12) is the explicit form of this action.

Keep in mind, however, that is a *weak-field* limit, because it is based on the approximation $A_{\Sigma;\mathrm{T}}\approx A_{\Sigma,\mathrm{T}}$, hence $g_{5\Sigma}=\overline{\kappa}h_{5\Sigma}\approx 2\overline{\kappa}A_\Sigma$, see (8.4). Thinking carefully about this approximation, we realize that the term $g_{5\Sigma}$, and *not* A_Σ , is to be is associated with the *exact* $\mathfrak{L}_{QED}$. The usual $\mathfrak{L}_{QED}$ is itself the weak field limit, that is, $\hbar c^2\mathfrak{L}_{QED}\approx\left(-\frac{1}{4}F^{\sigma\tau}F_{\sigma\tau}+A_\mu j^\mu\right)$. If we carefully backtrack, we can now deduce find the exact $\mathfrak{L}_{QED}$, for all field strengths, as follows:

First, since $g_{55}=g^{55}=\pm 1$, for a timelike (+) and spacelike (-) fifth dimension, respectively, we can rewrite (8.10) as $R=\frac{1}{2}\overline{\kappa}^2 g_{55}F^{\sigma\tau}F_{\sigma\tau}$. Now, we employ this expression in (7.12), and (8.10) in (7.14). Then we multiply (7.12) by g_{55} to obtain, after reduction, including $g_{55}g_{55}=1$, the following alternative expressions:

$$g_{55}R_{(5)}=-\tfrac{1}{2}\overline{\kappa}^2F^{\sigma\tau}F_{\sigma\tau}-\overline{\kappa}g_{55}j^5\text{, and} \tag{8.13}$$

$$g_{55}R_{(5)}=-\tfrac{1}{2}\overline{\kappa}^2F^{\sigma\tau}F_{\sigma\tau}+\overline{\kappa}g_{5\mu}j^\mu\,. \tag{8.14}$$

Because these are equivalent expressions for $g_{55}R_{(5)}$, we can equate these to deduce that $g_{5\mu}j^{\mu}=-g_{55}j^{5}$, or, in five-covariant form:

$$g_{5\Sigma}j^{\Sigma}=0\,. \tag{8.15}$$

This the direct relation which tied together the two method we used to obtain $R_{(5)}$.

Next, we return to (8.11). The terms with $A_{\mu}j^{\mu}$ were based on the $A_{\Sigma;\mathrm{T}}\approx A_{\Sigma,\mathrm{T}}$ approximation. The exact relation in (8.11), is that given geometrically by $4\kappa c\mathfrak{L}_{QED}=g_{55}R_{(5)}$. Converting back via $\kappa=\frac{1}{2}\hbar c\overline{\kappa}^{2}=8\pi G/c^{4}$, and using (8.13) and (8.14) with some term manipulation, we can now write the *exact* QED Lagrangian for all fields weak and strong, as:

$$\overline{\kappa}\hbar c^{2}\mathfrak{L}_{QED}=-\tfrac{1}{4}\overline{\kappa}F^{\sigma\tau}F_{\sigma\tau}+\tfrac{1}{2}g_{5\mu}j^{\mu}=-\tfrac{1}{4}\overline{\kappa}F^{\sigma\tau}F_{\sigma\tau}-\tfrac{1}{2}g_{55}j^{5}=\tfrac{1}{2\overline{\kappa}}g_{55}R_{(5)} \tag{8.16}$$

Now, we return to (8.3) which is an exact expression. In the weak field limit, where $A_{\Sigma;\mathrm{T}}\approx A_{\Sigma,\mathrm{T}}$, we may make the approximate substitution $g_{5\Sigma}=\overline{\kappa}h_{5\Sigma}\approx 2\overline{\kappa}A_{\Sigma}$ of (8.4) into the above, thus:

$$\overline{\kappa}\hbar c^{2}\mathfrak{L}_{QED}=-\tfrac{1}{4}\overline{\kappa}F^{\sigma\tau}F_{\sigma\tau}+\tfrac{1}{2}g_{5\mu}j^{\mu}\approx-\tfrac{1}{4}\overline{\kappa}F^{\sigma\tau}F_{\sigma\tau}+\overline{\kappa}A_{\mu}j^{\mu} \tag{8.17}$$

which recovers the customary QED Lagrangian. The action (8.12) contains an approximation symbol, because the $\mathfrak{L}_{QED}$ was taken to be the weak field $-\frac{1}{4}\overline{\kappa}F^{\sigma\tau}F_{\sigma\tau}+\overline{\kappa}A_{\mu}j^{\mu}$. If we now use (8.17), then the action (8.12) now becomes the exact expression, with $\hbar=c=1$:

$$\begin{aligned}S(g_{5\mu})&=g_{55}\frac{1}{4\kappa}\int R_{(5)}\sqrt{-g}\,d^{4}x=\int\mathfrak{L}_{QED}\sqrt{-g}\,d^{4}x\\&=\int\left(-\frac{1}{4}F^{\sigma\tau}F_{\sigma\tau}+\frac{1}{2\overline{\kappa}}g_{5\mu}j^{\mu}\right)\sqrt{-g}\,d^{4}x\approx\int\left(-\frac{1}{4}F^{\sigma\tau}F_{\sigma\tau}+A_{\mu}j^{\mu}\right)\sqrt{-g}\,d^{4}x\end{aligned}. \tag{8.18}$$

where in the final set of terms, we have employed the weak field $g_{5\Sigma}=\overline{\kappa}h_{5\Sigma}\approx 2\overline{\kappa}A_{\Sigma}$. The $g_{55}=\pm1$ simply represents the sign to be used depending on whether the fifth dimension is timelike or spacelike. Once a choice is, made, this is either a plus or a minus sign.

9. Electrodynamic Energy Tensors, including the Maxwell Stress Energy

To be added.

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