Dear Friends:

I'd like to lay out a nifty little mathematical calculation which allows a "decomposition" of the intrinsic spin matrices $s^i = \frac{1}{2}\hbar\sigma^i$ to include the position and momentum operators $x^i$, $p^i$, $i = 1,2,3$. To simplify matters, we will employ a Minkowski metric tensor with $\mathrm{diag}(\eta_{\mu\nu}) = (-1,+1,+1,+1)$ so that raising and lowering the space indexes $i = 1,2,3$ is simple and at will, and does not entail any sign reversal.

We start with the general cross product for two three-vectors **A** and **B**. Written in covariant (index) notation:

$$(\mathbf{A}\times\mathbf{B})_i \equiv \varepsilon_{ijk} A^j B^k . \quad (1)$$

One can easily confirm this by taking, for example, $(\mathbf{A}\times\mathbf{B})_3 \equiv A^1B^2 - A^2B^1$. Now, let's take the triple cross product $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$. We can apply (1) to itself using $(\mathbf{A}\times\mathbf{B})^j \equiv \varepsilon^{jmn} A_m B_n$, to write:

$$[(\mathbf{A}\times\mathbf{B})\times\mathbf{C}]_i = \varepsilon_{ijk}(\mathbf{A}\times\mathbf{B})^j C^k = \varepsilon_{ijk}\varepsilon^{jmn} A_m B_n C^k . \quad (2)$$

The fact that the crossing of **A** and **B** takes precedence over crossing with **C** is retained in the fact that $A_m B_n$ sum with $\varepsilon^{jmn}$, while $C^k$ alone sums into $\varepsilon_{ijk}$.

Let us now expand (2) for the component equation for which $i = 3$. The calculation is as such:

$$\begin{aligned}
[(\mathbf{A}\times\mathbf{B})\times\mathbf{C}]_3 &= \varepsilon_{3jk}\varepsilon^{jmn} A_m B_n C^k \\
&= \varepsilon_{312}\varepsilon^{123} A_2 B_3 C^2 + \varepsilon_{312}\varepsilon^{132} A_3 B_2 C^2 + \varepsilon_{321}\varepsilon^{231} A_3 B_1 C^1 + \varepsilon_{321}\varepsilon^{213} A_1 B_3 C^1 \\
&= A_1 B_3 C^1 + A_2 B_3 C^2 - A_3 B_1 C^1 - A_3 B_2 C^2 \\
&= A_1 B_3 C^1 + A_2 B_3 C^2 + A_3 B_3 C^3 - A_3 B_1 C^1 - A_3 B_2 C^2 - A_3 B_3 C^3 \\
&= A_1 B_3 C^1 + A_2 B_3 C^2 + A_3 B_3 C^3 - A_3(\mathbf{B}\cdot\mathbf{C})
\end{aligned} , \quad (3)$$

where we have added $0 = A_3 B_3 C^3 - A_3 B_3 C^3$ to the fourth line. Now in the final line, we hit an impasse, because $B_3$ is sandwiched between the terms we would like to form into the other dot product $\mathbf{A}\cdot\mathbf{C}$. In order to complete the calculation, we must make an assumption that the $A_i$ commute with $B_3$, i.e., that $[A_i, B_3] = 0$. For now, let us make this assumption.

Therefore, we carry out the commutation in (3), and continue along to write:

$$\begin{aligned}
[(\mathbf{A}\times\mathbf{B})\times\mathbf{C}]_3 &= \varepsilon_{3jk}\varepsilon^{jmn} A_m B_n C^k = A_1 B_3 C^1 + A_2 B_3 C^2 + A_3 B_3 C^3 - A_3(\mathbf{B}\cdot\mathbf{C}) \\
&= B_3(\mathbf{A}\cdot\mathbf{C}) - A_3(\mathbf{B}\cdot\mathbf{C}) = B_3 A_j C^j - A_3 B_j C^j
\end{aligned} . \quad (4)$$

Generalizing fully, we may now write (4) in two equivalent ways as:

$$\begin{cases} (\mathbf{A}\times\mathbf{B})\times\mathbf{C} = -\mathbf{A}(\mathbf{B}\cdot\mathbf{C}) + \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) \\ \varepsilon_{ijk}\varepsilon^{jmn}A_m B_n C^k = -A_i B_j C^j + B_i A_j C^j \end{cases} . \quad (5)$$

The reader will observe the well-known formula for the cross product.

Now, let's take the cross product in which $\mathbf{A} = \mathbf{x}$, $\mathbf{B} = \mathbf{p}$ and $\mathbf{C} = \boldsymbol{\sigma}$, where **x** is the position operator about the center of mass, **p** is the momentum operator, and $\boldsymbol{\sigma}$ are the Pauli spin matrices. We also take into account the Heisenberg canonical commutation relationship between the position and momentum operators, that is, $[x_\mu, p_\nu] = i\hbar\delta_{\mu\nu}$. This means that we will have to be careful at the juncture between equations (3) and (4), because the position and momentum operators along the same dimension do not commute.

So, we return to (3) with $\mathbf{A} = \mathbf{x}$, $\mathbf{B} = \mathbf{p}$ and $\mathbf{C} = \boldsymbol{\sigma}$, to write:

$$[(\mathbf{x}\times\mathbf{p})\times\boldsymbol{\sigma}]_3 = \varepsilon_{3jk}\varepsilon^{jmn}x_m p_n \sigma^k = x_1 p_3 \sigma^1 + x_2 p_3 \sigma^2 + x_3 p_3 \sigma^3 - x_3(\mathbf{p}\cdot\boldsymbol{\sigma}). \quad (6)$$

To take the next step, we want to place in $p_3$ front of the $x_i$. In so doing, we can commute $p_3$ with $x_i$ for $i = 1,2$. But, for $i = 3$, we must employ $x_3 p_3 = p_3 x_3 + i\hbar$. Therefore, (6) now becomes:

$$\begin{aligned} [(\mathbf{x}\times\mathbf{p})\times\boldsymbol{\sigma}]_3 &= \varepsilon_{3jk}\varepsilon^{jmn}x_m p_n \sigma^k = p_3 x_1 \sigma^1 + p_3 x_2 \sigma^2 + (p_3 x_3 + i\hbar)\sigma^3 - x_3(\mathbf{p}\cdot\boldsymbol{\sigma}) \\ &= p_3(\mathbf{x}\cdot\boldsymbol{\sigma}) - x_3(\mathbf{p}\cdot\boldsymbol{\sigma}) + i\hbar\sigma_3 = p_3 x_j \sigma^j - x_3 p_j \sigma^j + i\hbar\sigma_3 \end{aligned} , \quad (7)$$

lowering the index on $i\hbar\sigma^3$ with $\mathrm{diag}(\eta_{ij}) = (+1,+1,+1)$. Now all of a sudden, $i\hbar\sigma^3$ has made an unexpected appearance. Generalizing (7), we may write:

$$\begin{cases} [(\mathbf{x}\times\mathbf{p})\times\boldsymbol{\sigma}] = -\mathbf{x}(\mathbf{p}\cdot\boldsymbol{\sigma}) + \mathbf{p}(\mathbf{x}\cdot\boldsymbol{\sigma}) + i\hbar\boldsymbol{\sigma} \\ \varepsilon_{ijk}\varepsilon^{jmn}x_m p_n \sigma^k = -x_i p_j \sigma^j + p_i x_j \sigma^j + i\hbar\sigma_i \end{cases} , \quad (8)$$

This is the also the well-known formula for the triple-cross product, but with an additional term $i\hbar\boldsymbol{\sigma}$ emerging from the canonical commutation relationship. In fact, moving terms, equation (8) gives us a way to decompose the intrinsic spin matrix so as to contain the position and momentum, and as we shall also see, orbital angular momentum operators.

First, we rewrite (8) as:

$$\begin{cases} i\hbar\mathbf{s} = [(\mathbf{x}\times\mathbf{p})\times\mathbf{s}] + \mathbf{x}(\mathbf{p}\cdot\mathbf{s}) - \mathbf{p}(\mathbf{x}\cdot\mathbf{s}) \\ i\hbar s_i = \varepsilon_{ijk}\varepsilon^{jmn}x_m p_n s^k + x_i p_j s^j - p_i x_j s^j \end{cases} , \quad (9)$$

where we have multiplied through by $\frac{1}{2}\hbar$ and then set $s_i \equiv \frac{1}{2}\hbar\sigma_i$. This decomposes the intrinsic spin matrix into an expression involving itself, as well as the position and momentum operators.

Now, using the definition (1) but with $\mathbf{A} = \mathbf{x}$ and $\mathbf{B} = \mathbf{p}$, let's introduce the orbital angular momentum operator :

$$\mathbf{l}^j \equiv (\mathbf{x}\times\mathbf{p})^j \equiv l^j \equiv \varepsilon^{jmn} x_m p_n \quad (10)$$

It is easy to see, for example, that $l^3 = x_1 p_2 - x_2 p_1$. Using (10), we now rewrite (9) as:

$$\begin{cases} i\hbar\mathbf{s} = (\mathbf{l}\times\mathbf{s}) + \mathbf{x}(\mathbf{p}\cdot\mathbf{s}) - \mathbf{p}(\mathbf{x}\cdot\mathbf{s}) \\ i\hbar s_i = \varepsilon_{ijk} l^j s^k + x_i p_j s^j - p_i x_j s^j \end{cases}, \quad (11)$$

We see that part of this decomposition includes the cross-product $\mathbf{l}\times\mathbf{s}$ of the orbital angular momentum with the intrinsic spin. We may also multiply the lower equation (11) through by $\varepsilon^{mni}$ and then employ the commutation relationship $[s^m, s^n] = i\hbar\varepsilon^{mni} s_i$, to write:

$$[s^m, s^n] = \varepsilon^{mni}\varepsilon_{ijk} l^j s^k + \varepsilon^{mni} x_i p_j s^j - \varepsilon^{mni} p_i x_j s^j = l^m s^n + \varepsilon^{mni} x_i p_j s^j - \varepsilon^{mni} p_i x_j s^j . \quad (12)$$

Note, we have also made use of $\varepsilon^{mni}\varepsilon_{ijk} = \delta^{mni}{}_{ijk}$ .

Equation (11) allows us to decompose the total spin $\mathbf{S}$ for a Dirac field $\psi$, as follows:

$$\begin{cases} \mathbf{S} = \int (\overline{\psi}\mathbf{s}\psi) d^3x = -\frac{i}{\hbar}\int (\overline{\psi}[(\mathbf{l}\times\mathbf{s}) + \mathbf{x}(\mathbf{p}\cdot\mathbf{s}) - \mathbf{p}(\mathbf{x}\cdot\mathbf{s})]\psi) d^3x \\ S_i = \int (\overline{\psi} s_i \psi) d^3x = -\frac{i}{\hbar}\int (\overline{\psi}[\varepsilon_{ijk} l^j s^k + x_i p_j s^j - p_i x_j s^j]\psi) d^3x \end{cases} \quad (13)$$

See Ohanian, H., *What is Spin*, at http://jayryablon.wordpress.com/files/2008/04/ohanian-what-is-spin.pdf, equation (18).

Two questions: Does this seem correct? Has anybody seen this before?

Best regards,

Jay R. Yablon, April 18, 2008